Higher Pro-Arrows:
Towards a Model for Naturality Pretype Theory

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HoTTEST
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https://anuyts.github.io/files/2024/natpt-hottest-pres.pdf

Introduction

Why I care

Not out of an intrinsic interest in

- (directed) algebraic topology,
- synthetic (∞, ∞)-category theory.

Consequences

- Types stratified by finite dimensions. (Cf. Haskell but less weird.)
- I'm not afraid of strict equality. I am afraid of coherence obligations.
- I don't mind if my model doesn't present spaces. But I want it to compute!
- Factorization systems are not my native language.

I want better languages for **verified functional programming**!

Programs should be categorically structured.

With native support for relations/morphisms/isomorphisms:

- Parametricity for free!
- Functoriality for free!
- Naturality for free!
- Variance of dependent multi-argument functions sorted out for free!

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So how is **Directed TT** relevant to **verified functional programming**?

An example problem

$$
\begin{tikzcd}
 & & M & N & & \mathrm{Monad} \\
A \arrow[d, "f"'] & \mathrm{List}\,A & \mathrm{WriterT}\,(\mathrm{List}\,A)\,M & \mathrm{WriterT}\,(\mathrm{List}\,A)\,N & & \\
B & \mathrm{List}\,B & \mathrm{WriterT}\,(\mathrm{List}\,B)\,M & \mathrm{WriterT}\,(\mathrm{List}\,B)\,N & & \\
\pitchfork & & & & \mathrm{Monad} & \\
\mathrm{Type} & \mathrm{Monoid} & & & & \times
\end{tikzcd}
$$

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\begin{tikzcd}
 & & M & N & & \mathrm{Monad} \\
A \arrow[d, "f"'] & \mathrm{List}\,A & \mathrm{WriterT}\,(\mathrm{List}\,A)\,M & \mathrm{WriterT}\,(\mathrm{List}\,A)\,N & & \\
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\end{tikzcd}
$$

$$M \xrightarrow{g} N \quad \in \quad \mathsf{Monad}$$

$$\begin{array}{cccc}
A & \mathsf{List}\,A & \mathsf{WriterT}\,(\mathsf{List}\,A)\,M & \mathsf{WriterT}\,(\mathsf{List}\,A)\,N \\
\downarrow f & \downarrow \mathsf{List}\,f & & \\
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\end{array}$$

$$\mathsf{Type} \xrightarrow{\mathsf{List}} \mathsf{Monoid} \qquad \times$$

$$
\begin{tikzcd}
 & & M \arrow[r, "g"] & N & \in & \mathrm{Monad} \arrow[dddd, no head] \\
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\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & & & \mathsf{Monad} & \\
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In plain DTT

Functoriality of $\mathrm{List} : \mathrm{Type} \to \mathrm{Monoid}$:

- Object action: $(\mathrm{List}\, A, [], ++)$
- Functorial action:
  - $\mathrm{List}\, f : \mathrm{List}\, A \to \mathrm{List}\, B$ **(by recursion)**
  - $\mathrm{List}\, f$ is a monoid morphism:
    - $\mathrm{List}\, f$ preserves $[]$ (trivial)
    - $\mathrm{List}\, f$ preserves $++$ **(by induction)**

  + functor laws **(by induction)**

Functoriality of
$\mathrm{WriterT} : \mathrm{Monoid} \to \mathrm{MonadTrans}$

- Object action: $\mathrm{WriterT}\, W \in \mathrm{MonadTrans}$
  - Object action: $\mathrm{WriterT}\, W\, M \in \mathrm{Monad}$
    - Object action: Define $\mathrm{WriterT}\, W\, M\, A$
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      + functor laws
    - return & bind + naturality

... Object action: $\mathrm{WriterT}\, W \in \mathrm{MonadTrans}$

  - Functorial action $\mathrm{WriterT}\, W\, g$
    - Respects return & bind

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  - $\mathrm{lift} : M \to \mathrm{WriterT}\, W\, M$ + naturality
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In parametric DTT

Functoriality of List : Type $\to$ Monoid:

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$$
\begin{tikzcd}
 & & M \arrow[r, "g", "\cong"'] & N & \in & \mathrm{Monad} \arrow[ddddd, no head] \\
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 & & & & \mathrm{Monad} & \\
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\end{tikzcd}
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$$
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B & \mathsf{List}\,B & \mathsf{WriterT}\,(\mathsf{List}\,B)\,M \arrow[r, equals, bend right] & \mathsf{WriterT}\,(\mathsf{List}\,B)\,N \arrow[dr, phantom, "\in"] & & \\
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In HoTT (assuming f, g and $h = \mathrm{List}\, f$ are isos)

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- 🎁 Functorial action:
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  + 🎁 functor laws

Variance and modalities

WriterT $W\,M\,A := M(A \times W)$ is **covariant** w.r.t.

- W : Monoid
- M : Monad
- A : Type

ReaderT $R\,M\,A := R \to M A$ is **contravariant** w.r.t.

- R : Type

return : $A \to$ WriterT $W\,M\,A$ is **natural** w.r.t.

- W : Monoid
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Ignoring variance

- HoTT: only consider **isomorphisms**
  ☹ Not everything is an isomorphism.
- Param'ty: **relations**, not morphisms
  ☹ Don't know how to compute fmap.

Naturality TT

- Preserve isomorphisms
- Preserve relations
- Keep track of action on morphisms

Hence:

- Use functoriality/naturality when possible
- Use HoTT when applicable
- Use param'ty when necessary

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- HoTT: only consider **isomorphisms**
  ☹ Not everything is an isomorphism.
- Param'ty: **relations**, not morphisms
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Naturality TT

- Preserve isomorphisms
- Preserve relations
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Hence:

- Use functoriality/naturality when possible
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Variance and modalities

WriterT $W\,M\,A := M(A \times W)$ is **covariant** w.r.t.

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ReaderT $R\,M\,A := R \to M A$ is **contravariant** w.r.t.

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Pretypes: A Note on Fibrancy

A note on fibrancy

A **presheaf model** of DTT can account for:

- The **existence** of shapes
  (point, path, morphism, bridge, ...)
- **Unary operations** on shapes (src, rfl)
- **Unary equations** on shapes
  ($\mathrm{src} \circ \mathrm{rfl} = \mathrm{id}$)

Fibrancy allows for:

- Other arities (composition, ...)
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Naturality **Pre**type Theory

We **ignore** fibrancy for now:

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- There are promising techniques for defining fibrancy internally:
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(Aside) Actually, I'd like your feedback

Definition

A CwF is **locally democratic** if every arrow $\sigma : \Delta \to \Gamma$ is isomorphic to some $\pi : \Gamma.T \to \Gamma$.

Internalizing an AWFS [§8.5 of my PhD thesis]

- A CwF is exactly a model of the **structural rules** of DTT.
- On a **locally democratic CwF**, the following **correspond**:
  - Defining an **AWFS** whose right replacement monad RR **preserves pullbacks**,
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Model-first Approach

Separation of concerns:

We need **modalities** to keep track of **variance**.

- ➔ Instantiate **MTT** (Multimodal Type Theory) [GKNB21]
- ☺ The syntax is **their problem**!

We need **substructural intervals** for **bridges** / **morphisms** / **paths**.

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(Back to NatPT)
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MTT (Multimod[e/al] Type Theory)

MTT [GKNB21] is parametrized by a **2-category** called the **mode theory**:

- modes $p, q, r, \ldots$
- modalities $\mu : p \to q$

$$\frac{\Gamma\ \mathsf{ctx} \ @\ q}{\Gamma, 🔒_\mu\ \mathsf{ctx}\ @\ p} \qquad \frac{\Gamma, 🔒_\mu \vdash T\ \mathsf{type}\ @\ p}{\Gamma \vdash \langle \mu \mid T \rangle\ \mathsf{type}\ @\ q} \qquad \frac{\Gamma, 🔒_\mu \vdash t : T\ @\ p}{\Gamma \vdash \mathsf{mod}_\mu\, t : \langle \mu \mid T \rangle\ @\ q}$$

- (2-cells $\alpha : \mu \Rightarrow \nu$).

Semantics:

- $[\![p]\!]$ is a (often presheaf) category modelling all of DTT,
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$$\begin{array}{ccccccc} \exists(u:\mathbb{U}) & \dashv & \Finv[u] & \dashv & \forall(u:\mathbb{U}) & \dashv & \between[u] \\ \Sigma(u:\mathbb{U}) & \dashv & \Omega[u] & \dashv & \Pi(u:\mathbb{U}) & & \end{array}$$

as modalities.

Problem: They bind / depend on variables.
(Not supported by MTT.)

Solution: Put **shape context** Ξ in the **mode**.

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$$\begin{array}{ccccccc} [\![\text{🔒}^{\exists u}_{\Finv u}]\!] & \dashv & [\![\text{🔒}^{\Finv u}_{\forall u}]\!] & \dashv & [\![\text{🔒}^{\forall u}_{\between u}]\!] & & \\ & & [\![\Finv u]\!] & \dashv & [\![\forall u]\!] & \dashv & [\![\between u]\!] \end{array}$$

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$$
\begin{array}{ccccccccc}
(\exists_{\mathbb{U}}^{\int \Xi})_! & \dashv & (\exists_{\mathbb{U}}^{\int \Xi})^* & \dashv & (\exists_{\mathbb{U}}^{\int \Xi})_* & & \\
 & & \| \wr & & \| \wr & & \\
 & & (\Finv_{\mathbb{U}}^{\int \Xi})_! & \dashv & (\Finv_{\mathbb{U}}^{\int \Xi})^* & \dashv & (\Finv_{\mathbb{U}}^{\int \Xi})_*
\end{array}
$$

$$
\exists_{\mathbb{U}}^{\Xi} \dashv \Finv_{\mathbb{U}}^{\Xi} \dashv \forall_{\mathbb{U}}^{\Xi} \dashv \between_{\mathbb{U}}^{\Xi}
$$

$$
\begin{array}{ccccccc}
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 & & [\![\Finv u]\!] & \dashv & [\![\forall u]\!] & \dashv & [\![\between u]\!]
\end{array}
$$

Idea: Treat

$$
\begin{array}{ccccccc}
\exists(u : \mathbb{U}) & \dashv & \reflectbox{$\exists$}[u] & \dashv & \forall(u : \mathbb{U}) & \dashv & \between[u] \\
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$$
\begin{array}{ccccccccc}
(\exists^{\int \Xi}_{\mathbb{U}})_! & \dashv & (\exists^{\int \Xi}_{\mathbb{U}})^* & \dashv & (\exists^{\int \Xi}_{\mathbb{U}})_* & & & & \\
 & & \| \wr & & \| \wr & & & & \\
 & & (\reflectbox{$\exists$}^{\int \Xi}_{\mathbb{U}})_! & \dashv & (\reflectbox{$\exists$}^{\int \Xi}_{\mathbb{U}})^* & \dashv & (\reflectbox{$\exists$}^{\int \Xi}_{\mathbb{U}})_* & &
\end{array}
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$$

$$
\begin{array}{ccccccc}
[\![\text{🔒}^{\exists u}_{\reflectbox{$\exists$} u}]\!] & \dashv & [\![\text{🔒}^{\reflectbox{$\exists$} u}_{\forall u}]\!] & \dashv & [\![\text{🔒}^{\forall u}_{\between u}]\!] & & \\
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$$\exists^{\Xi}_{\mathbb{U}} \dashv Ⅎ^{\Xi}_{\mathbb{U}} \dashv \forall^{\Xi}_{\mathbb{U}} \dashv ≬^{\Xi}_{\mathbb{U}}$$

$$\begin{array}{ccccccc} [\![🔒^{\exists u}_{Ⅎ u}]\!] & \dashv & [\![🔒^{Ⅎ u}_{\forall u}]\!] & \dashv & [\![🔒^{\forall u}_{≬ u}]\!] & & \\ & & [\![Ⅎ u]\!] & \dashv & [\![\forall u]\!] & \dashv & [\![≬ u]\!] \end{array}$$

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 & & \| \wr & & \| \wr & & \\
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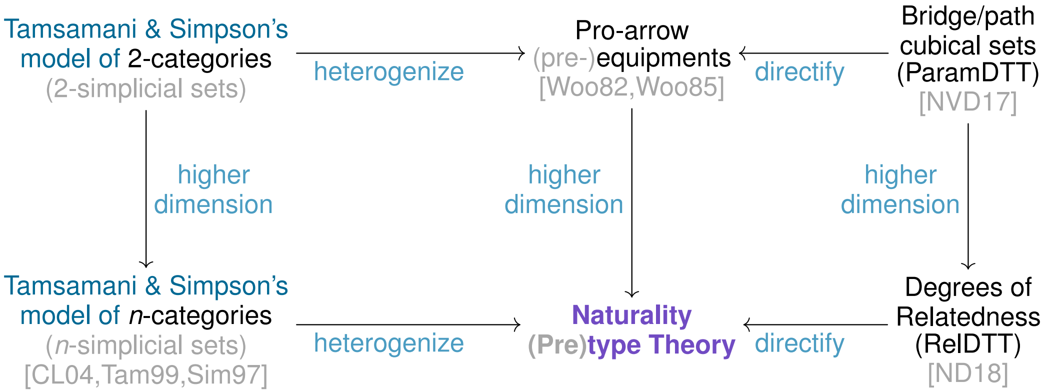
\end{array}
$$

Introduction: Wrapping up

- We want to preserve **relations**, **morphisms** *and* **isomorphisms**.
- We need **variance** ➔ MTT
- We need **intervals** ➔ MTraS
- We need **fibrancy** ➔ future work (internal)
- For now, we care about:
  - a **mode theory**,
  - a **presheaf model** for each mode,
  - an **adjunction** for each modality,
  - a **functor** for each interval.

Three Approaches to the Model



Tamsamani & Simpson's model of n-Categories

Tamsamani (1999)
Simpson (1997)
see Cheng & Lauda (2004)

A **reflexive graph** Γ has:

- A set of **nodes** Γ_0
- A set of **edges** Γ_1
- $\Gamma_{\mathrm{src}}, \Gamma_{\mathrm{tgt}} : \Gamma_1 \to \Gamma_0$ and $\Gamma_{\mathrm{rfl}} : \Gamma_0 \to \Gamma_1$

It is a diagram in Set:

$$\begin{tikzcd} \Gamma_0 \arrow[r, "\Gamma_{\mathrm{rfl}}"] & \Gamma_1 \arrow[l, bend right=50, "\Gamma_{\mathrm{src}}"'] \arrow[l, bend left=50, "\Gamma_{\mathrm{tgt}}"] \end{tikzcd}$$

A **simplicial set** Γ has:

- For each n, a set of ***n*-simplices** Γ_n (nodes, edges, triangles, tetrahedra, ...)
- For each monotonic $f : \{0..m\} \hookrightarrow \{0..n\}$, a **face map** $\Gamma_f : \Gamma_n \to \Gamma_m$ (vertices of, edges of, faces of, ...)
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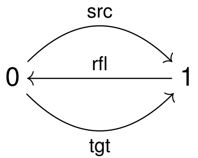

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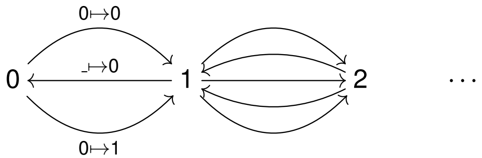

Simplex category Δ

Δ is a skeleton of NonEmptyFinLinOrd

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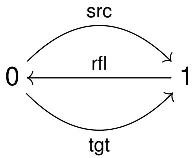

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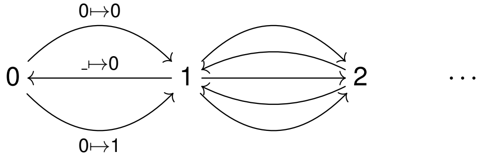

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Nerve $N(\mathscr{C})$ of a category $\mathscr{C}$

Simplicial set whose:

- **nodes** are objects
- **edges** are morphisms
- **triangles** are commutative diagrams
- $(n \geq 3)$**-simplices** uniquely exist

Segal condition

Q: When is a simplicial set the nerve of a category?

A: If every chain of n edges

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \bullet$$

is the **spine** (Hamiltonian path) of a unique n-simplex. I.e. if compositions uniquely exist.

$$\text{Categories} \quad \simeq \quad \text{Segal simplicial sets}$$

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Let $(\mathscr{V}, I, \otimes)$ be a **monoidal category**.

A $\mathscr{V}$-enriched category $\mathscr{C}$ has:

- A (big) set of **objects**
- For each $x, y \in \mathrm{Obj}(\mathscr{C})$, a **Hom-thing** $\mathrm{Hom}(x, y) \in \mathrm{Obj}(\mathscr{V})$,
- $\mathrm{id}_x : I \to \mathrm{Hom}(x, x)$
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Strict n-category

- A 0-category is a **set**.
- An $(n+1)$-category is a **category enriched over n-categories**.

Q: Can we understand higher categories via simplicial sets?

Cheng & Lauda's Guidebook: [CL04]
A thousand times **yes!**

Tamsamani & Simpson: [Sim97,Tam99]
One such time **yes!**
➔ using **double** / **n-fold** categories

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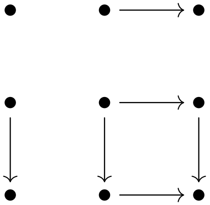
A **double category** $\mathscr{C}$ has:

- objects
- horiz. arrows / (1)-arrows (1-cells)
- vertical arrows / (2)-arrows (trivial)
- squares (2-cells)

and can be defined as a **bisimplicial set** $\mathscr{C} \in \mathrm{Psh}(\Delta \times \Delta)$ satisfying the **Segal condition** in each dimension.

A **T&S 2-category** is:

a **double category** whose **vertical arrows** are **trivial**.



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$$
\begin{tikzcd} \bullet \end{tikzcd}
\qquad
\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}
$$

$$
\begin{tikzcd} \bullet \arrow[d] \\ \bullet \end{tikzcd}
\qquad
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$$\begin{tikzcd} \bullet \arrow[d] \\ \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

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$$\begin{tikzcd} \bullet & \bullet \arrow[r] & \bullet \\ \bullet \arrow[d, equal] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[d, equal] \\ \bullet & \bullet \arrow[r] & \bullet \end{tikzcd}$$

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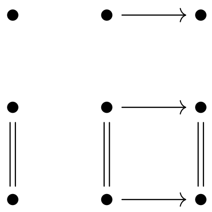

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A **T&S *n*-category** is:

an ***n*-fold** category where only

- (1)-arrows, (1-cells)
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- (1,2,3)-cubes, (3-cells)
- ...

can be non-trivial.

Pretypes!

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Pretypes!

Three Approaches to the Model

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Tamsamani \& Simpson's model of 2-categories (2-simplicial sets)} \arrow[r, "\text{heterogenize}"'] \arrow[d, "\text{higher dimension}"] & \text{Pro-arrow (pre-)equipments [Woo82,Woo85]} \arrow[d, "\text{higher dimension}"'] & \text{Bridge/path cubical sets (ParamDTT) [NVD17]} \arrow[l, "\text{directify}"] \arrow[d, "\text{higher dimension}"'] \\
\text{Tamsamani \& Simpson's model of } n\text{-categories (}n\text{-simplicial sets) [CL04,Tam99,Sim97]} \arrow[r, "\text{heterogenize}"'] & \textbf{Naturality (Pre)type Theory} & \text{Degrees of Relatedness (RelDTT) [ND18]} \arrow[l, "\text{directify}"]
\end{tikzcd}
$$

Pro-arrow Equipments

Richard J. Wood (1982, 1985)

(Pro-arrow) Equipment

An **equipment** $\mathscr{C}$ is a **double category** with

- objects
- arrows ($\to$)
- pro-arrows ($\nrightarrow$)
- squares

such that every arrow $\varphi : x \to y$ has **"graph"** pro-arrows

$$\varphi^{\ddagger} : x \nrightarrow y, \qquad \varphi^{\dagger} : y \nrightarrow x$$

such that $(\ldots)$.

Example (Set)

Set is an **equipment** with:

- sets
- functions
- relations
  - identity relation: equality
  - $(R;S)(x,z) =$ $\exists y . R(x,y) \wedge S(y,z)$
- proofs that $R(a,b) \Rightarrow S(f\,a, g\,b)$

$$\begin{tikzcd}
A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] & B \arrow[d, "g"] \\
C \arrow[r, "S"', "\shortmid"{marking}] & D
\end{tikzcd}$$

(Pro-arrow) Equipment

An **equipment** $\mathscr{C}$ is a **double category** with

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- arrows ($\to$)
- pro-arrows ($\not\to$)
- squares

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Cat is an **equipment** with:

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Cat is a **T&S 2-category** with:

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- nat. transformations $\overset{\text{end}}{\forall} a.\mathrm{Hom}(F\,a, G\,a)$

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$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mathrm{Hom}", "\shortmid" marking] \arrow[d, "F"'] & \mathscr{A} \arrow[d, "G"] \\ \mathscr{C} \arrow[r, "\mathrm{Hom}"', "\shortmid" marking] & \mathscr{C} \end{tikzcd}$$

To get **heterogeneous** nat. transformations:
drop T&S's triviality condition!

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$$\begin{tikzcd} \mathscr{A} \arrow[r, "\mathscr{P}", "\shortmid"{marking}] \arrow[d, "F"'] & \mathscr{B} \arrow[d, "G"] \\ \mathscr{C} \arrow[r, "\mathscr{Q}"', "\shortmid"{marking}] & \mathscr{D} \end{tikzcd}$$

Example (Cat)

Cat is a **T&S 2-category** with:

- categories
- functors
- trivial (2)-arrows
- nat. transformations $\overset{\text{end}}{\forall} a.\mathrm{Hom}(F\,a, G\,a)$
  i.e. $\overset{\text{end}}{\forall} a, b.\mathrm{Hom}(a, b) \Rightarrow \mathrm{Hom}(F\,a, G\,b)$

$$
\begin{tikzcd}
\mathscr{A} \arrow[r, "\mathrm{Hom}", "|" marking] \arrow[d, "F"'] & \mathscr{A} \arrow[d, "G"] \\
\mathscr{C} \arrow[r, "\mathrm{Hom}"', "|" marking] & \mathscr{C}
\end{tikzcd}
$$

To get **heterogeneous** nat. transformations:
drop T&S's triviality condition!

Example (Cat)

Cat is an **equipment** with:

- categories
- functors
- profunctors $\mathscr{P} : \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathrm{Set}$
  - identity profunctor: Hom
  - $(\mathscr{P}; \mathscr{Q})(x, z) =$
    $\overset{\text{coend}}{\exists} y.\mathscr{P}(x, y) \times \mathscr{Q}(y, z)$
- $\overset{\text{end}}{\forall} a, b.\mathscr{P}(a, b) \Rightarrow \mathscr{Q}(F\,a, G\,b)$

$$
\begin{tikzcd}
\mathscr{A} \arrow[r, "\mathscr{P}", "|" marking] \arrow[d, "F"'] & \mathscr{B} \arrow[d, "G"] \\
\mathscr{C} \arrow[r, "\mathscr{Q}"', "|" marking] & \mathscr{D}
\end{tikzcd}
$$

Three Approaches to the Model

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{Tamsamani \& Simpson's}\\ \text{model of 2-categories}\\ \text{(2-simplicial sets)}\end{array} \arrow[r, "\text{heterogenize}"'] \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"] & \begin{array}{c}\text{Pro-arrow}\\ \text{(pre-)equipments}\\ \text{[Woo82,Woo85]}\end{array} \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"'] & \begin{array}{c}\text{Bridge/path}\\ \text{cubical sets}\\ \text{(ParamDTT)}\\ \text{[NVD17]}\end{array} \arrow[l, "\text{directify}"] \arrow[d, "\begin{array}{c}\text{higher}\\ \text{dimension}\end{array}"'] \\
\begin{array}{c}\text{Tamsamani \& Simpson's}\\ \text{model of } n\text{-categories}\\ (n\text{-simplicial sets})\\ \text{[CL04,Tam99,Sim97]}\end{array} \arrow[r, "\text{heterogenize}"'] & \begin{array}{c}\textbf{Naturality}\\ \textbf{(Pre)type Theory}\end{array} & \begin{array}{c}\text{Degrees of}\\ \text{Relatedness}\\ \text{(RelDTT)}\\ \text{[ND18]}\end{array} \arrow[l, "\text{directify}"]
\end{tikzcd}$$

Higher

Pro-arrow

Equipments

Higher Pro-arrow Equipments

Set is . . .

- ☹ A large set
- 😐 A category
- ☺ An equipment

Cat is . . .

- ☹ A category
- 😐 A 2-category
- ☺ An equipment

Eqmnt is . . .

- ☹ An equipment
- ☺ A 2-equipment

Eqmnt has:

Objects	Equipments
Arrows	Equipment functors
Pro-arrows	Equipment profunctors: Contain arrows and pro-arrows
Pro-pro-arrows	Equipment **pro-profunctors:** Contain pro-arrows
Squares	. . .
Cubes	. . .

Higher Equipment

An n-**equipment** is an n-**fold category** (. . .)

$\Rightarrow \mathscr{C} \in \mathrm{Psh}(\Delta^n_{\dagger,\ddagger})$

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For any category $\mathscr{W}$,
$\mathrm{Psh}(\mathscr{W})$ models **DTT, with a universe** U^{HS}**.**

Let $W \in \mathrm{Obj}(\mathscr{W})$.
A W-cell of U^{HS} contains:

- a notion of dependent W-cells

Looking at this differently

Define $\dot{\mathscr{W}} :\cong \mathscr{W}$.
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☺ **Directed layer** on top of your favorite TT!

In particular:

		$\mathrm{Psh}(\top)$	(sets)
$\mathsf{U}^{\mathsf{dir}}_{\top}$	$\in$	$\mathrm{Psh}(\Delta)$	(categories)
$\mathsf{U}^{\mathsf{dir}}_{\Delta}$	$\in$	$\mathrm{Psh}(\dot{\Delta} \times \Delta)$	(eqmnts)
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Against self-classification

By nature, classifiers (typically) do **NOT** contain themselves:

- **All of mankind** is **not** an example of a **human**.
- **The world's literature** is **not** an example of a **book**.

Forcing things to be otherwise is (a priori) **unreasonable**.

Classifiers of **collection-like** objects:

- **Set** is **more than** a (large) set.
- **Cat** is **more than** a (large) category.

It's not because you **can truncate** to achieve **self-classification**, that you **should!**

➔ Provide the user with the **unscathed classifier** *and* the **truncation modality**.
➔ Use **multimode** type theory.

☺ **Fixpoints:** ∞**Grpd** is a (large) ∞-groupoid.

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. . . and while I am ranting . . .

Against the Grothendieck Construction as "the" Σ-type for categories

Grothendieck Construction

Given a **category** $\mathscr{C}$ and
a **functor** $\mathscr{D} : \mathscr{C} \to \mathrm{Arws}(\mathrm{Cat})$,
i.e. **eqmnt functor** $\mathscr{H} : \mathrm{FPro}(\mathscr{C}) \to \mathrm{Cat}$,
the category $\int_{\mathscr{C}} \mathscr{D}$ has:

- objects $(c, d \in \mathscr{D}(c))$
- morphisms

$$\left(c_1 \xrightarrow{\gamma} c_2, \mathscr{D}(\gamma)(d_1) \xrightarrow{\delta} d_2\right)$$

$\mathrm{Arws}(\mathrm{Cat}) \in \mathrm{Cat}$ is **truncated**.

$$\mathrm{FPro} \dashv \mathrm{Arws} : \mathrm{Eqmnt} \to \mathrm{Cat}$$

Arws	Discards pro-arrows
FPro	Freely adds "graph" pro-arrows
Pros	Discards arrows

Let's generalize from $\mathrm{FPro}(\mathscr{C})$ to $\mathscr{E} \in \mathrm{Eqmnt}$.

$$\int_{\mathscr{C}} \mathscr{D} \qquad \qquad \mathrm{Pros}\left(\oint_{\mathrm{FPro}(\mathscr{C})} \mathscr{H}\right)$$

$$\mathscr{C} \qquad \qquad \mathrm{Pros}(\mathrm{FPro}(\mathscr{C}))$$

$$\mathrm{Pros}\left(\oint_{\mathrm{FPro}(\mathscr{C})} \mathscr{H}\right) \xrightarrow{\mathrm{Pros}(\mathrm{Fst})} \mathrm{Pros}(\mathrm{FPro}(\mathscr{C}))$$

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Three Approaches to the Model

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\text{Tamsamani \& Simpson's model of 2-categories (2-simplicial sets)} \arrow[r, "\text{heterogenize}"'] \arrow[d, "\text{higher dimension}"] & \text{Pro-arrow (pre-)equipments [Woo82,Woo85]} \arrow[d, "\text{higher dimension}"'] & \text{Bridge/path cubical sets (ParamDTT) [NVD17]} \arrow[l, "\text{directify}"] \arrow[d, "\text{higher dimension}"'] \\
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Degrees of Relatedness
(RelDTT)

Nuyts and Devriese (2018) @ LICS

- **Relational version** of what NatTT intends to be
- Perhaps **alienating**:
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  - Much less than higher category theory
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- Equip types with **multiple**, **proof-relevant relations** $s \frown_i t$ indexed by **degree** i:
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- **Reflexivity:** $(a : A) \frown^A_i (a : A)$
  (Semantically, prop. eq. = def. eq.)
- **Degradation:** $((a : A) \frown^R_i (b : B)) \to ((a : A) \frown^R_{i+1} (b : B))$
- **Dependency:** $(a : A) \frown^R_i (b : B)$ **presumes** $R : A \frown^{\mathrm{U}}_{i+1} B$
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Understanding degrees

$(a : A) \frown_0^A (b : A)$	Equality.
$(a : A) \frown_0^R (b : B)$	Heterogeneous equality along ...
$R : (A : \mathrm{U}^0) \frown_1^{\mathrm{U}^0} (B : \mathrm{U}^0)$	Any relation R.
$P : (G : \mathrm{Grp}) \frown_1^{\mathrm{Grp}} (H : \mathrm{Grp})$	Any logical/algebraic relation P.
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Understanding modalities: Parametricity

par : types → values

$$\mathit{if} : (\mathbf{par} \mid X : \mathrm{U}^0) \to B\,X$$

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X = Y \arrow[r] \arrow[d] & \mathit{if}_X = \mathit{if}_Y \arrow[dd] \\
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\top \arrow[r] & \top
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$$B : \mathrm{U}^0 \to \mathrm{U}^0$$

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The Mode Theory

- Modes are **depths** $p \in \mathbb{Z}_{\geq -1}$
- Modalities $\mu : p \to q$ are
  functions $\{0 \leq \ldots \leq q\} \to \{(=) \leq 0 \leq \ldots \leq p \leq \top\} : i \mapsto i \cdot \mu$
  where $f : (\mu \mid x : A) \to B(x)$ sends
  $$(r : x \frown^{A}_{i \cdot \mu} y) \quad \to \quad f(x) \frown^{B(r)}_{i} f(y).$$
  Modal types:
  $$\mathrm{mod}_\mu\, x \frown^{\langle \mu | A \rangle}_{i} \mathrm{mod}_\mu\, y \quad = \quad x \frown^{A}_{i \cdot \mu} y$$
- 2-cells are degree-wise inequalities.

Depth p is modelled in cubical sets with $p + 1$ different dimension flavours.

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Three Approaches to the Model

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Tamsamani \& Simpson's model of 2-categories (2-simplicial sets)} \arrow[r, "\text{heterogenize}"'] \arrow[d, "\text{higher dimension}"] & \text{Pro-arrow (pre-)equipments [Woo82,Woo85]} \arrow[d, "\text{higher dimension}"'] & \text{Bridge/path cubical sets (ParamDTT) [NVD17]} \arrow[l, "\text{directify}"] \arrow[d, "\text{higher dimension}"'] \\
\text{Tamsamani \& Simpson's model of } n\text{-categories (}n\text{-simplicial sets) [CL04,Tam99,Sim97]} \arrow[r, "\text{heterogenize}"'] & \textbf{Naturality (Pre)type Theory} & \text{Degrees of Relatedness (RelDTT) [ND18]} \arrow[l, "\text{directify}"]
\end{tikzcd}
$$

Higher Pro-arrows:
Directifying
Degrees of Relatedness

- Equip types with **multiple, proof-relevant relations** $s \twoheadrightarrow_i t$ indexed by **degree** i
  - Proofs called i**-jets** (pro^{i-1}-arrows).
- Describe **function behaviour** by saying how functions **influence degree** and **orientation** of jets.

Degrees of Relatedness: **Four Laws**

- **Reflexivity:** $(a = b : A) \to ((a : A) \frown^{A}_{i} (b : A))$
  (Semantically, prop. eq. = def. eq.)
- **Degradation:** $((a : A) \frown^{R}_{i} (b : B)) \to ((a : A) \frown^{R}_{i+1} (b : B))$
- **Dependency:** $(a : A) \frown^{R}_{i} (b : B)$ **presumes** $R : A \frown^{\mathsf{U}}_{i+1} B$
- **Identity extension:** $(a : A) \frown^{A}_{0} (b : A)$ means $a = b : A$.
  $\rightsquigarrow$ heterogeneous $\frown_0$ serves as heterogeneous equality.

Pretypes!

Higher equipments: **Three Laws**

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Understanding degrees

$(a : A) \rightarrowtriangle^{f}_{0} (b : B)$	a maps to b along ...
$f : (A : \mathsf{U}^0) \rightarrowtriangle^{\mathsf{U}^0}_{1} (B : \mathsf{U}^0)$	Any function f.
$\varphi : (G : \mathsf{Grp}) \rightarrowtriangle^{\mathsf{Grp}}_{1} (H : \mathsf{Grp})$	Any morphism φ.
$\psi : (G : \mathsf{Grp}) \rightarrowtriangle^{\mathscr{P}}_{1} (M : \mathsf{Monoid})$	Any heterogeneous morphism ψ along ...
$\mathscr{P} : (\mathsf{Grp} : \mathsf{U}^1) \rightarrowtriangle^{\mathsf{U}^1}_{2} (\mathsf{Monoid} : \mathsf{U}^1)$	Any profunctor $\mathscr{P}$ e.g. $\mathscr{P} = \mathrm{Hom}_{\mathsf{Monoid}}(U_{\mathsf{Grp}}\sqcup, \sqcup)$

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e.g. $\mathscr{P} = \mathrm{Hom}_{\mathrm{Monoid}}(U_{\mathrm{Grp}} \sqcup, \sqcup)$

Understanding degrees

$(a : A) \rightarrowtriangle_0^{f} (b : B)$ — a maps to b along ...

$f : (A : \mathrm{U}^0) \rightarrowtriangle_1^{\mathrm{U}^0} (B : \mathrm{U}^0)$ — Any function f.

$\varphi : (G : \mathrm{Grp}) \rightarrowtriangle_1^{\mathrm{Grp}} (H : \mathrm{Grp})$ — Any morphism φ.

$\psi : (G : \mathrm{Grp}) \rightarrowtriangle_1^{\mathscr{P}} (M : \mathrm{Monoid})$ — Any heterogeneous morphism ψ along ...

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e.g. $\mathscr{P} = \mathrm{Hom}_{\mathrm{Monoid}}(U_{\mathrm{Grp}} \sqcup, \sqcup)$

Understanding degrees

$(a : A) \rightarrowtriangle^{f}_{0} (b : B)$	a maps to b along …
$f : (A : \mathrm{U}^0) \rightarrowtriangle^{\mathrm{U}^0}_{1} (B : \mathrm{U}^0)$	Any function f.
$\varphi : (G : \mathrm{Grp}) \rightarrowtriangle^{\mathrm{Grp}}_{1} (H : \mathrm{Grp})$	Any morphism φ.
$\psi : (G : \mathrm{Grp}) \rightarrowtriangle^{\mathscr{P}}_{1} (M : \mathrm{Monoid})$	Any heterogeneous morphism ψ along …
$\mathscr{P} : (\mathrm{Grp} : \mathrm{U}^1) \rightarrowtriangle^{\mathrm{U}^1}_{2} (\mathrm{Monoid} : \mathrm{U}^1)$	Any profunctor $\mathscr{P}$ e.g. $\mathscr{P} = \mathrm{Hom}_{\mathrm{Monoid}}(U_{\mathrm{Grp}} \sqcup, \sqcup)$

Understanding modalities: Limits

$\mathbf{lim}^{\oplus}$

$$e : (\mathbf{lim}^{\oplus} \mid X : \mathrm{Grp}) \to |X|$$

$$\begin{tikzcd}
G = H \arrow[rr] \arrow[d] & & e_G = e_H \arrow[dd] \\
p : G \xrightarrow[0]{\mathrm{id}_{\mathrm{Grp}}} H \arrow[d, "\ddagger"'] & & \\
\varphi : G \xrightarrow[1]{\mathrm{Hom}_{\mathrm{Grp}}} H \arrow[rr] \arrow[d] & & e_G \xrightarrow[0]{|\varphi|} e_H \arrow[d] \\
\top \arrow[rr] & & \top
\end{tikzcd}$$

$\mathbf{ftr}^{\oplus}$

$$|\sqcup| : (\mathbf{ftr}^{\oplus} \mid \mathrm{Grp}) \to \mathrm{U}^0$$

$$\begin{tikzcd}
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G \xrightarrow[1]{\mathrm{Hom}_{\mathrm{Grp}}} H \arrow[r, dotted, no head] \arrow[d] & {|G| \xrightarrow[1]{\mathrm{Hom}_{\mathrm{U}^0}} |H|} \arrow[d] \\
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G \overset{\mathrm{Hom}_{\mathrm{Grp}}}{\rightarrow\!\!\vartriangleright_1} H \arrow[r, dotted, no head] \arrow[d] & {|G| \overset{\mathrm{Hom}_{\mathrm{U}^0}}{\rightarrow\!\!\vartriangleright_1} |H|} \arrow[d] \\
\top \arrow[r] & \top
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\top \arrow[r] & \top
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$$\begin{tikzcd}
G = H \arrow[r] \arrow[d] & {|G| = |H|} \arrow[d] \\
G \xrightarrow[0]{\mathrm{id}_{\mathsf{Grp}}} H \arrow[r, dotted, dash] \arrow[d, "\ddagger"'] & {|G| \xrightarrow[0]{\mathrm{id}_{\mathsf{U}^0}} |H|} \arrow[d, "\ddagger"] \\
G \xrightarrow[1]{\mathrm{Hom}_{\mathsf{Grp}}} H \arrow[r] \arrow[d] & {|G| \xrightarrow[1]{\mathrm{Hom}_{\mathsf{U}^0}} |H|} \arrow[d] \\
\top \arrow[r] & \top
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\top \arrow[r] & \top
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G \overset{\mathrm{Hom}_{\mathrm{Grp}}}{\rightarrowtriangle}_1 H \arrow[r] \arrow[d] & |G| \overset{\mathrm{Hom}_{\mathrm{U}^0}}{\rightarrowtriangle}_1 |H| \arrow[d] \\
\top \arrow[r] & \top
\end{tikzcd}$$

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$$\mathrm{hd} : (\mathbf{lim}^{\ominus} \mid X : \mathrm{Coalg}_{\mathbb{N}\times\sqcup}) \to |X| \to \mathbb{N}$$

$$\begin{tikzcd}
X = Y \arrow[rr] \arrow[d] & & hd_X = hd_Y \arrow[dd] \\
p : X \xrightarrow[0]{\mathrm{id}_{\mathrm{Coalg}_{\mathbb{N}\times\sqcup}}} Y \arrow[d, "\ddagger"'] & & \\
\varphi : X \xrightarrow[1]{\mathrm{Hom}_{\mathrm{Coalg}_{\mathbb{N}\times\sqcup}}} Y \arrow[r, "\ominus"] \arrow[d] & hd_X \xleftarrow[0]{\sqcup\circ|\varphi|} hd_Y \arrow[d] & \\
\top \arrow[r] & \top &
\end{tikzcd}$$

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$$\lambda X.(|X| \to \mathbb{N}) : (\mathbf{ftr}^{\ominus} \mid \mathrm{Coalg}_{\mathbb{N}\times\sqcup}) \to \mathrm{U}^0$$

$$\begin{tikzcd}
X = Y \arrow[r] \arrow[d] & \mathbb{N}^{|X|} = \mathbb{N}^{|Y|} \arrow[d] \\
X \xrightarrow[0]{\mathrm{id}_{\mathrm{Coalg}_{\mathbb{N}\times\sqcup}}} Y \arrow[r, dotted, no head] \arrow[d, "\ddagger"'] & \mathbb{N}^{|X|} \xrightarrow[0]{\mathrm{id}_{\mathrm{U}^0}} \mathbb{N}^{|Y|} \arrow[d, "\dagger"] \\
G \xrightarrow[1]{\mathrm{Hom}_{\mathrm{Grp}}} H \arrow[r, dotted, no head] \arrow[d] & \mathbb{N}^{|X|} \xleftarrow[1]{\mathrm{Hom}_{\mathrm{U}^0}} \mathbb{N}^{|Y|} \arrow[d] \\
\top \arrow[r] & \top
\end{tikzcd}$$

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G \overset{\mathrm{Hom}_{\mathrm{Grp}}}{\twoheadrightarrow_1} H \arrow[rr, dotted, no head] \arrow[d] & & \mathbb{N}^{|X|} \overset{\mathrm{Hom}_{\mathrm{U}^0}}{\twoheadleftarrow_1} \mathbb{N}^{|Y|} \arrow[d] \\
\top \arrow[rr] & & \top
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p : X \rightharpoondown_0^{\mathrm{id}_{\mathsf{Coalg}_{\mathbb{N}\times\sqcup}}} Y \arrow[d, "\ddagger"'] & \\
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\top \arrow[r] & \top
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G \xrightarrow{\mathrm{Hom}_{\mathsf{Grp}}}_1 H \arrow[r, "\ominus"] \arrow[d] & \mathbb{N}^{|X|} \xleftarrow{\mathrm{Hom}_{\mathsf{U}^0}}_1 \mathbb{N}^{|Y|} \arrow[d] \\
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\end{tikzcd}$$

The mode theory

NatPT instantiates MTT (Multimode Type Theory) with:

- Modes are **dimensions** $p \in \mathbb{N}$ (+ you can mark a degree $i < n$ as symmetric)
- Modalities $\mu : p \to q$ are certain functions
  $\{0, \dots, q-1\} \to \{(=), 0, \dots, p-1, \top\} \times \{\circledast, \oplus, \ominus, \otimes\}$
  where $f : (\mu \mid x : A) \to B(x)$ sends

$$(r : x \rightarrow^{A}_{i \cdot \mu} y) \quad \to \quad f(x) \rightarrow^{B(r)}_{i} f(y).$$

  Modal types:

$$\mathrm{mod}_{\mu}\, x \rightarrow^{\langle \mu | A \rangle}_{i} \mathrm{mod}_{\mu}\, y \quad = \quad x \rightarrow^{A}_{i \cdot \mu} y$$

- 2-cells are degree-wise inequalities.

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Three Approaches to the Model

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Tamsamani \& Simpson's model of 2-categories (2-simplicial sets)} \arrow[r, "\text{heterogenize}"'] \arrow[d, "\text{higher dimension}"] & \text{Pro-arrow (pre-)equipments [Woo82,Woo85]} \arrow[d, "\text{higher dimension}"'] & \text{Bridge/path cubical sets (ParamDTT) [NVD17]} \arrow[l, "\text{directify}"] \arrow[d, "\text{higher dimension}"'] \\
\text{Tamsamani \& Simpson's model of } n\text{-categories (}n\text{-simplicial sets) [CL04,Tam99,Sim97]} \arrow[r, "\text{heterogenize}"'] & \textbf{Naturality (Pre)type Theory} & \text{Degrees of Relatedness (RelDTT) [ND18]} \arrow[l, "\text{directify}"]
\end{tikzcd}
$$

The Model

The Twisted Prism Functor

Δ is a skeleton of (hence $\simeq$) NEFinLinOrd.

Twisted Prism Functor [PK20]

$\sqcup \ltimes \mathbb{I} : \mathrm{NEFinLinOrd} \to \mathrm{NEFinLinOrd} : W \mapsto W^{\mathrm{op}} \uplus_{<} W$

$$a \longrightarrow b \quad \mapsto \quad \begin{tikzcd} (a,0) \arrow[d] & (b,0) \arrow[l] \arrow[d] \\ (a,1) \arrow[r] & (b,1) \end{tikzcd}$$

MTraS shape modelled by $\sqcup \ltimes \mathbb{I}$ reconciles:

- Hom as a **contra-/covariant bifunctor**,
- Hom as a **constrained function type**.

$\mathbb{I}$ as an MTraS-shape is better behaved on $\bowtie$:

Twisted Cube Category $\bowtie$ [PK20]

(Roughly) the subcategory of NEFinLinOrd (or Δ) generated by $\top$ and $\sqcup \ltimes \mathbb{I}$.

➔ Use $\bowtie$ instead of Δ.

ℹ Pinyo & Kraus carve $\bowtie$ out of **graph** category.

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$$a \longrightarrow b \qquad \mapsto \qquad \begin{array}{ccc} (a,0) & \longleftarrow & (b,0) \\ \downarrow & & \downarrow \\ (a,1) & \longrightarrow & (b,1) \end{array}$$

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- Hom as a **contra-/covariant bifunctor**,
- Hom as a **constrained function type**.

$\mathbb{I}$ as an MTraS-shape is better behaved on $\bowtie$:

Twisted Cube Category $\bowtie$ [PK20]

(Roughly) the subcategory of NEFinLinOrd (or Δ) generated by $\top$ and $\sqcup \ltimes \mathbb{I}$.

➔ Use $\bowtie$ instead of Δ.

ℹ Pinyo & Kraus carve $\bowtie$ out of **graph** category.

The Twisted Prism Functor

Δ is a skeleton of (hence $\simeq$) NEFinLinOrd.

Twisted Prism Functor [PK20]

$\sqcup \ltimes \mathbb{I} : \mathrm{NEFinLinOrd} \to \mathrm{NEFinLinOrd} :$
$W \mapsto W^{\mathrm{op}} \uplus_{<} W$

$$a \longrightarrow b \quad \mapsto \quad \begin{tikzcd} (a,0) \arrow[d] & (b,0) \arrow[l] \arrow[d] \\ (a,1) \arrow[r] & (b,1) \end{tikzcd}$$

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Jet Set of dimension n

Set equipped with n Prop-valued **jet-relations** $\rightarrowtriangle_i$ such that:

- $\rightarrowtriangle_i$ is **reflexive**
- $\rightarrowtriangle_i$ implies $\leftrightarrowtriangle_{i+1}$

- **Intervals** $(\!|\rightarrowtriangle_i|\!) = \{0 \rightarrowtriangle_i 1\}$
- **Twisted prism** functor $\sqcup \ltimes (\!|\rightarrowtriangle_i|\!)$ only **op**s degree i
- **Jet cubes** are generated by $\top$ and $\sqcup \ltimes (\!|\rightarrowtriangle_i|\!)$
  - What is a morphism of jet cubes?

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Jet Set of dimension n

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- $\multimapdotinv_i$ implies $\multimapbothvert_{i+1}$

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- **Twisted prism** functor $\sqcup \ltimes (\!|\multimapdotinv_i|\!)$ only **op**s degree i
- **Jet cubes** are generated by $\top$ and $\sqcup \ltimes (\!|\multimapdotinv_i|\!)$
  - What is a morphism of jet cubes?

The Category of **Jet Cubes**

$$\begin{tikzcd}
\mathrm{JetCube}_n \arrow[rr, bend left] \arrow[dr, dotted, "\mathrm{ForgetDeg}"'] & & \mathrm{JetSet}_n \arrow[d, "U"] \\
& \mathrm{Cube}^{⌑}_M \arrow[r, dotted, "\mathrm{Endpoints}"'] & \mathrm{Set}
\end{tikzcd}$$

- What **interval operations** do you want? ➔ $\mathrm{Cube}^{\boxslash}_M \cong \mathrm{Kleisli}(M)^{\mathrm{op}}$
- Do you want diagonals? ➔ $⌑ \in \{\Box, \boxslash\}$
- Turns out only $\mathrm{Cube}^{\Box}_{0,1,\neg}$ and $\mathrm{Cube}^{\Box}_{\mathrm{FreeBoolAlg}}$ really work.

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The Category of **Jet Cubes**

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& \mathsf{Cube}^{\text{⌑}}_M \arrow[r, "\mathsf{Endpoints}"'] & \mathsf{Set}
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& \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{JetSet}_n \arrow[d, "U"] \\
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When is a morphism of **cubes** a morphism of **jet cubes**?

$$\dfrac{\textsc{Terminal}}{\vdash () : V \to ()}$$

$$\textsc{Src:Fwd}\quad \dfrac{\vdash \varphi : V \to \mathsf{Op}_i^{\boxdot}(W)}{\vdash (\varphi, 0/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimap_i \rrparenthesis)} \qquad \textsc{Src:Bck}\quad \dfrac{\vdash \varphi : V \to \mathsf{Op}_i^{\boxdot}(W)}{\vdash (\varphi, 1/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimapinv_i \rrparenthesis)}$$

$$\textsc{Tgt:Fwd}\quad \dfrac{\vdash \varphi : V \to W}{\vdash (\varphi, 1/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimap_i \rrparenthesis)} \qquad \textsc{Tgt:Bck}\quad \dfrac{\vdash \varphi : V \to W}{\vdash (\varphi, 0/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimapinv_i \rrparenthesis)}$$

$$\textsc{Inv:Fwd}\quad \dfrac{\vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimapinv_i \rrparenthesis)}{\vdash (\varphi, \neg t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimap_i \rrparenthesis)} \qquad \textsc{Inv:Bck}\quad \dfrac{\vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimap_i \rrparenthesis)}{\vdash (\varphi, \neg t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \multimapinv_i \rrparenthesis)}$$

$$\textsc{Prism:Fwd}\quad \dfrac{\vdash \varphi : V \to W}{\vdash (\varphi, \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis \multimap_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis \multimap_i \rrparenthesis)} \qquad \textsc{Prism:Bck}\quad \dfrac{\vdash \varphi : V \to W}{\vdash (\varphi, \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis \multimapinv_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis \multimapinv_i \rrparenthesis)}$$

$$\textsc{Symmetrize}\quad \dfrac{\vdash \varphi : \mathsf{FSym}_i^{\boxdot} V \to W}{\vdash \varphi : V \to \mathsf{USym}_i^{\boxdot} W}$$

$$\textsc{Wkn}\quad \dfrac{\vdash \varphi : \mathsf{SymCl}_i^{\boxdot}(V) \to W \qquad R \in \{\multimap, \multimapinv, \multimapboth\}}{\vdash (\varphi, \mathbf{i}/\oslash) : (V, \mathbf{i} : \llparenthesis R_i \rrparenthesis) \to W}$$

$$\textsc{Exchange}\quad \dfrac{\vdash \varphi : (V, \mathbf{j} : \llparenthesis \multimapboth_i \rrparenthesis, U_1, \mathbf{i} : \llparenthesis \multimapboth_i \rrparenthesis, U_2) \to W}{\vdash \varphi : (V, \mathbf{i} : \llparenthesis \multimapboth_i \rrparenthesis, U_1, \mathbf{j} : \llparenthesis \multimapboth_i \rrparenthesis, U_2) \to W}$$

$$\textsc{Concursor}\quad \dfrac{P \in \{\multimap, \multimapinv, \multimapboth\} \qquad Q \in \{\multimap, \multimapinv\} \qquad j > i \qquad \vdash \varphi : \mathsf{SymCl}_i^{\boxdot}(\mathsf{SymCl}_j^{\boxdot} U, V) \to W}{\vdash (\varphi, \mathbf{j}/\mathbf{i}) : (U, \mathbf{j} : \llparenthesis P_j \rrparenthesis, V) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}$$

$$\textsc{Conn:Prism:Src-Neutral}\quad \dfrac{(Q, \Diamond) \in \{(\multimap, \vee), (\multimapinv, \wedge)\} \qquad \vdash \varphi : \mathsf{SymCl}_i^{\boxdot} V \to W \qquad \vdash (\varphi, t/\mathbf{i}) : \mathsf{Op}_i^{\boxdot} V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}{\vdash (\varphi, t \Diamond \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\ \text{Boo}$$

$$\textsc{Conn:Prism:Tgt-Neutral}\quad \dfrac{(Q, \Diamond) \in \{(\multimap, \wedge), (\multimapinv, \vee)\} \qquad \vdash \varphi : \mathsf{SymCl}_i^{\boxdot} V \to W \qquad \vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}{\vdash (\varphi, t \Diamond \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\ \text{Boo}$$

$$\textsc{Conn:Prism-Inv:Src-Neutral}\quad \dfrac{(Q, \Diamond, P) \in \{(\multimap, \vee, \multimapinv), (\multimapinv, \wedge, \multimap)\} \qquad \vdash \varphi : \mathsf{SymCl}_i^{\boxdot} V \to W \qquad \vdash (\varphi, t/\mathbf{i}) : \mathsf{Op}_i^{\boxdot} V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}{\vdash (\varphi, t \Diamond \neg\mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis P_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\ \text{Boo}$$

$$\textsc{Conn:Prism-Inv:Tgt-Neutral}\quad \dfrac{(Q, \Diamond, P) \in \{(\multimap, \wedge, \multimapinv), (\multimapinv, \vee, \multimap)\} \qquad \vdash \varphi : \mathsf{SymCl}_i^{\boxdot} V \to W \qquad \vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}{\vdash (\varphi, t \Diamond \neg\mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis P_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\ \text{Boo}$$

$$\textsc{Conn:Degree-Symmetric}\quad \dfrac{Q \in \{\multimap, \multimapinv\} \qquad \Diamond \in \{\vee, \wedge\} \qquad \vdash (\varphi, s/\mathbf{i}) : \mathsf{SymCl}_i^{\boxdot} V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \qquad \vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}{\vdash (\varphi, t \Diamond s/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\ \text{Boo}$$

Semantic Modalities

⌛ In progress…

Three Approaches to the Model

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Tamsamani \& Simpson's model of 2-categories (2-simplicial sets)} \arrow[r, "\text{heterogenize}"'] \arrow[d, "\text{higher dimension}"] & \text{Pro-arrow (pre-)equipments [Woo82,Woo85]} \arrow[d, "\text{higher dimension}"'] & \text{Bridge/path cubical sets (ParamDTT) [NVD17]} \arrow[l, "\text{directify}"] \arrow[d, "\text{higher dimension}"'] \\
\text{Tamsamani \& Simpson's model of } n\text{-categories (}n\text{-simplicial sets) [CL04,Tam99,Sim97]} \arrow[r, "\text{heterogenize}"'] & \textbf{Naturality (Pre)type Theory} & \text{Degrees of Relatedness (RelDTT) [ND18]} \arrow[l, "\text{directify}"]
\end{tikzcd}
$$

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Thanks!

Questions?

Depth n types

- ***i*-edge** relations $\frown_i$
- **Dependency:**
  $r : a \frown^R_i b$ presumes $R : A \frown^{\mathrm{U}}_{i+1} B$
- **Degradation:**
  $a \frown_i b \Rightarrow a \frown_{i+1} b$
- Modalities change indices:

$$\begin{tikzcd}
a \frown_0 b \arrow[d] & fa \frown_0 fb \arrow[d] \\
a \frown_1 b \arrow[d] \arrow[ur] & fa \frown_1 fb \\
a \frown_2 b \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
a \frown_0 b \arrow[r] \arrow[d] & fa \frown_0 fb \arrow[d] \\
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\end{tikzcd}$$

$\boxed{\mathbf{lim}^{\oplus}}$ $\boxed{\mathbf{ftr}^{\oplus}}$

n-equipments

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- Modalities change indices & orientation:

$$\begin{tikzcd}
a \rightarrowtriangle_0 b \arrow[d, "\circledast"'] & fa \rightarrowtriangle_0 fb \arrow[d, "\circledast"] \\
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a \rightarrowtriangle_2 b \arrow[ur, "\oplus"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
a \rightarrowtriangle_0 b \arrow[r, "\oplus"] \arrow[d, "\circledast"'] & fa \rightarrowtriangle_0 fb \arrow[d, "\circledast"] \\
a \rightarrowtriangle_1 b \arrow[r, "\ominus"] \arrow[d, "\circledast"'] & fa \rightarrowtriangle_1 fb \arrow[d, "\circledast"] \\
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- **Companion / conjoint:**
  $(\ddagger, \dagger) : a \mathrel{-\!\triangleright}_i b \Rightarrow a \mathrel{\triangleleft\!\triangleright}_{i+1} b$
- Modalities change indices & orientation:

$$\begin{tikzcd}
a \mathrel{-\!\triangleright}_0 b \arrow[d, "\circledast"'] & fa \mathrel{-\!\triangleright}_0 fb \arrow[d, "\circledast"] \\
a \mathrel{-\!\triangleright}_1 b \arrow[d, "\circledast"'] \arrow[ur, "\oplus"] & fa \mathrel{-\!\triangleright}_1 fb \\
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a \frown_2 b \arrow[ur] &
\end{tikzcd}$$

$\boxed{\mathbf{lim}^{\oplus}}$

$$\begin{tikzcd}
a \frown_0 b \arrow[r] \arrow[d] & fa \frown_0 fb \arrow[d] \\
a \frown_1 b \arrow[r] \arrow[d] & fa \frown_1 fb \arrow[d] \\
a \frown_2 b \arrow[r] & fa \frown_2 fb
\end{tikzcd}$$

$\boxed{\mathbf{ftr}^{\oplus}}$

n-equipments

- ***i*-jet** (pro^{i-1}-arrow) relations $\mathrel{-\!\triangleright}_i$
- **Dependency:**
  $j : a \mathrel{-\!\triangleright}_i^{J} b$ presumes $J : A \mathrel{-\!\triangleright}_{i+1}^{\mathsf{U}} B$
- **Companion / conjoint:**
  $(\ddagger, \dagger) : a \mathrel{-\!\triangleright}_i b \Rightarrow a \mathrel{\triangleleft\!-\!\triangleright}_{i+1} b$
- Modalities change indices & orientation:

$$\begin{tikzcd}
a \mathrel{-\!\triangleright}_0 b \arrow[d, "\circledast"'] & fa \mathrel{-\!\triangleright}_0 fb \arrow[d, "\circledast"] \\
a \mathrel{-\!\triangleright}_1 b \arrow[d, "\circledast"'] \arrow[ur, "\oplus"] & fa \mathrel{-\!\triangleright}_1 fb \\
a \mathrel{-\!\triangleright}_2 b \arrow[ur, "\oplus"] &
\end{tikzcd}$$

$\boxed{\mathbf{lim}^{\ominus}}$

$$\begin{tikzcd}
a \mathrel{-\!\triangleright}_0 b \arrow[r, "\oplus"] \arrow[d, "\circledast"'] & fa \mathrel{-\!\triangleright}_0 fb \arrow[d, "\circledast"] \\
a \mathrel{-\!\triangleright}_1 b \arrow[r, "\ominus"] \arrow[d, "\circledast"'] & fa \mathrel{-\!\triangleright}_1 fb \arrow[d, "\circledast"] \\
a \mathrel{-\!\triangleright}_2 b \arrow[r, "\oplus"] & fa \mathrel{-\!\triangleright}_2 fb
\end{tikzcd}$$

$\boxed{\mathbf{ftr}^{\ominus}}$

Understanding modalities: Irrelevance

irr : values $\longrightarrow$ values

$$[\,] : (\mathbf{irr} \mid n : \mathbb{N}) \to \mathrm{List}_n\, A$$

shi : values $\longrightarrow$ types

$$\lambda n.\mathrm{List}_n\, A : (\mathbf{shi} \mid n : \mathbb{N}) \to \mathrm{U}^0$$

Understanding modalities: Irrelevance

irr : values $\to$ values

$$[\,] : (\mathbf{irr} \mid n : \mathbb{N}) \to \mathrm{List}_n\, A$$

$$\begin{tikzcd}
m = n \arrow[rr] \arrow[dd] & & {[\,]_m = [\,]_n} \arrow[dd] \\
& & \\
r : m \overset{\mathbb{N}}{\underset{0}{\frown}} n \arrow[d] & & {[\,]_m \overset{\mathrm{List}_\bullet\, A}{\underset{0}{\frown}} [\,]_n} \arrow[d] \\
\bullet : \top \arrow[urr] & & \top
\end{tikzcd}$$

shi : values $\to$ types

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$$\begin{tikzcd}
m = n \arrow[r] \arrow[dd] & \mathrm{List}_m\, A = \mathrm{List}_n\, A \arrow[d] \\
& \mathrm{List}_m\, A \overset{\mathrm{U}^0}{\underset{0}{\frown}} \mathrm{List}_n\, A \arrow[d] \\
m \overset{\mathbb{N}}{\underset{0}{\frown}} n \arrow[d] & \mathrm{List}_m\, A \overset{\mathrm{U}^0}{\underset{1}{\frown}} \mathrm{List}_n\, A \arrow[d] \\
\top & \top
\end{tikzcd}$$

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m \overset{\mathbb{N}}{\underset{0}{\frown}} n \arrow[d] \arrow[ur, dotted, dash] & \mathsf{List}_m\ A \overset{\mathsf{U}^0}{\underset{1}{\frown}} \mathsf{List}_n\ A \arrow[d] \\
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m \frown^{\mathbb{N}}_{0} n \arrow[d] \arrow[ur, dotted, no head] & \mathsf{List}_m\, A \frown^{\mathsf{U}^0}_{1} \mathsf{List}_n\, A \arrow[d] \\
\top \arrow[ur, dotted, no head] & \top
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Understanding modalities: Irrelevance

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m \frown^{\mathbb{N}}_{0} n \arrow[ur] \arrow[d] & \mathrm{List}_m\ A \frown^{\mathrm{U}^0}_{1} \mathrm{List}_n\ A \arrow[d] \\
\top \arrow[ur] & \top
\end{tikzcd}$$