Transpension for Cubes without Diagonals

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Introduction Several variants of dependent type theory have been modelled in presheaf models, and different authors have come up with various internal operators justified by either specific or general presheaf models of type theory. In earlier work [ND24], we introduced the transpension type, which is right adjoint to universal quantification over a shape (typically a representable object, such as a relational 'bridge' or homotopy 'path' interval); and we showed that transpension, in combination with two additional unsurprising operations (strictification and certain pushouts), is sufficiently expressive that we can recover many of these presheaf internalization operators found in the literature. This includes operators used for internal parametricity [BCM15, CH21], nominal type theory [PMD15], and to define a universe of Kan types internally [LOPS18].

However, we introduced the transpension type in a highly general form, where it did not necessarily commute with shape variable substitution. We handled this by expelling all aspects of shape variables and substitution from the syntax, confining these to the mode theory of the *Modal Transpension System* (MTraS), an (extended) instance of Multimodal Type Theory (MTT) [GKNB21]. As such, the generality of MTraS came with a price: the system is highly complex and frankly unpractical, and decidability of typing is a non-trivial problem at best. As such, MTraS and the transpension type have not seen widespread use.

We can approach the tradeoff between generality and simplicity differently, by restricting ourselves to settings where the transpension type *does* commute with shape substitution. This is the case for shapes $\mathbb{U}$ whose fresh weakening operation $\Finv_{\mathbb{U}} : \mathrm{Ty}(\Gamma) \to \mathrm{Ty}(\Gamma, u : \mathbb{U})$ is fully faithful [Nuy20, thm. 6.3.1], a technical condition that roughly corresponds to linear and affine shapes. From a syntactic viewpoint, this can be understood as follows: the transpension type is essentially a tool for intentional variable capture, which is not stable under variable contraction. In particular, this condition applies to the homotopy interval in Bezem, Coquand and Huber's (BCH) cubical model of HoTT [BCH14] and the relational interval in Bernardy, Coquand and Moulin's (BCM) model of internal parametricity [BCM15] and its combination with cubical HoTT (CH) [CH21]. For these settings, it should be possible to develop a Fully Faithful Transpension System (FFTraS) that is less general than MTraS, but is non-modal, treats substructural shape variables as part of the syntax and has a corresponding, sane, substitution calculus for both shape and other variables. In fact, this system has already been partially developed in an introductory section of [ND24, §2] and shown to pseudo-embed into MTraS when instantiated on an appropriately substructural shape. On the other hand, Riley's type theory with a tiny object [Ril24], which features a right adjoint to exponentiation by an appropriately substructural shape, should translate to FFTraS. However, FFTraS [ND24, §2] still attempts to cover a wide range of semantic situations, and as such leaves a number of questions regarding shape substitution unanswered. For this reason, we trade some more generality for simplicity and propose **TraSCwoD**: a Transpension System for Cubes without Diagonals. This system still supports the BCH, BCM and CH models.

Limitations of FFTraS Consider two of the most unusual rules of the system: the introduction rule of the transpension type, and the telescope quantification rule it relies upon:

$$\textsc{ff:ctx-forall}\quad \frac{\Gamma, u : \mathbb{U}, \delta : \Delta\ \mathsf{ctx} \qquad \text{No shape vars in } \Delta}{\Gamma, \forall u.(\delta : \Delta)\ \mathsf{ctx}} \qquad\qquad \textsc{ff:transp:intro}\quad \frac{\Gamma, \forall u.(\delta : \Delta) \vdash a : A}{\Gamma, u : \mathbb{U}, \delta : \Delta \vdash \mathsf{mer}[u]\, a : \between[u]\, A}$$

The idea here is that the transpension type is a dependent right adjoint to telescope quantification, and the introduction rule internalizes transposition of the adjunction. If we want our calculus to be closed under substitution, then the term $\mathsf{mer}[u]\, a : \between[u]\, A$ can be substituted with any $\sigma : \Theta \to (\Gamma, u : \mathbb{U}, \delta : \Delta)$. Now the intended semantics of FFTraS do not preclude situations where

$\mathbb{U}$ may support exchange with itself or with another shape, nor where $u : \mathbb{U}$ may be substituted by a non-variable term. This forces us to generalize the premise of FF:CTX-FORALL from contexts of the form $\Gamma, u : \mathbb{U}, \delta : \Delta$ (with no shape variables in Δ) to any context Θ *over* such one. Without further narrowing down the potential models of our system, it is hard to get a grasp of what such Θ may look like in general, let alone how to apply telescope quantification to it.

Cubes without Diagonals Recall that a semicartesian category is a monoidal category whose unit object is terminal. Assume a category of *shapes* $\mathcal{S}$ with a designated object $I \in \mathrm{Obj}(\mathcal{S})$ to which there are no non-identity arrows. We consider four categories $\mathcal{S}^{\otimes}, \mathcal{S}^{\otimes}_{\pi}, \mathcal{S}^{\otimes}_{\sigma}, \mathcal{S}^{\otimes}_{\pi,\sigma}$ generated by $\mathcal{S}$: the free [σ: symmetric] [π: semicartesian | no π: monoidal] category with unit I over $\mathcal{S}$.

TraSCwoD will take semantics in presheaves over a monoidal category $\mathcal{C}$ of one of the above forms, or more generally $\mathcal{C} \times \mathcal{W}$ for some other category $\mathcal{W}$. Context extension with a shape $\mathbb{U} \in \mathcal{S}$ ($\mathbb{U} \neq I$) will be modelled by Day convolution with $\mathbf{y}\mathbb{U}$. By consequence, our shapes will be substructural: 1. shape variable contraction is not allowed, 2. exchange of shape variables is possible for $\mathcal{S}^{\otimes}_{\sigma}$ and $\mathcal{S}^{\otimes}_{\pi,\sigma}$, 3. weakening over shape variables is possible for $\mathcal{S}^{\otimes}_{\pi}$ and $\mathcal{S}^{\otimes}_{\pi,\sigma}$. These semantics still comprise BCH, BCM (non-refined) and CH (w.r.t. the affine bridge interval).

TraSCwoD: A Transpension System for Cubes without Diagonals We start from a substitution calculus which is structural for normal variables, substructural for shape variables as described above, and allows exchange of normal and shape variables in one direction: $(\Gamma, x : A, u : \mathbb{U}) \to (\Gamma, u : \mathbb{U}, x : A)$, but only if weakening over shapes is allowed (otherwise A is not well-typed in context $\Gamma, u : \mathbb{U}$). The idea, besides soundness in the model, is that variables to the left of u are regarded as being fresh for u. Next, we add rules for shape quantification over $\mathbb{U}$, which is right adjoint to context extension with $\mathbb{U}$, which is itself a parametric right adjoint. This means that the functor $\exists\!\!\!\!\!-_{\mathbb{U}} : \mathrm{Psh}(\mathcal{C}) \to \mathrm{Psh}(\mathcal{C})/\mathbb{U} : \Gamma \mapsto ((\Gamma, u : \mathbb{U}), u)$ has a left adjoint $\exists_{\mathbb{U}}$. Thus, we can get our rules from Gratzer, Cavallo, Kavvos, Guatto and Birkedal [GCK+22]:

CWOD:FORALL
$$\frac{\Gamma, u : \mathbb{U} \vdash A\,\mathsf{type}}{\Gamma \vdash \forall\, u.A\,\mathsf{type}}$$

CWOD:FORALL:INTRO
$$\frac{\Gamma, u : \mathbb{U} \vdash a : A}{\Gamma \vdash \lambda u.a : \forall\, u.A}$$

CWOD:FORALL:ELIM
$$\frac{\Theta \vdash r : \mathbb{U} \qquad \exists_{\mathbb{U}}(\Gamma, r) \vdash f : \forall\, u.A}{\Theta \vdash f\, r : A[r/u]}$$

Internally, the way to access variables in the context $\exists_{\mathbb{U}}(\Theta, r)$ will be via a substitution $\exists_{\mathbb{U}}(\Theta, r) \to \Theta/r$, where Θ/r is the part of Θ that is fresh for r and is defined as follows: 1. $\Theta/r = \Theta$ if r is a constant, 2. $(\Gamma, v : \mathbb{V}, \Delta)/r = (\Gamma, \mathsf{shps}(\Delta))$ if r mentions variable v of shape $\mathbb{V} \in \mathcal{S}$ (and therefore no other shape variable, by construction of $\mathcal{C}$). Here, $\mathsf{shps}(\Delta)$ removes all normal variables from Δ, while its behaviour on shape variables depends on the structural rules.

Next, we introduce the transpension type, using the right adjoint $\exists\!\!\!\!\!-_{\mathbb{U}} \dashv \forall_{\mathbb{U}}$:

CWOD:TRANSP
$$\frac{\Theta \vdash r : \mathbb{U} \qquad \forall_{\mathbb{U}}(\Theta, r) \vdash A\,\mathsf{type}}{\Theta \vdash \between[r]\, A\,\mathsf{type}}$$

CWOD:TRANSP:INTRO
$$\frac{\Theta \vdash r : \mathbb{U} \qquad \forall_{\mathbb{U}}(\Theta, r) \vdash a : A}{\Theta \vdash \mathsf{mer}[r]\, a : \between[r]\, A}$$

CWOD:TRANSP:ELIM
$$\frac{\Gamma, u : \mathbb{U} \vdash t : \between[u]\, A}{\Gamma \vdash \mathsf{unmer}(u.t) : A}$$

Here, $\forall_{\mathbb{U}}(\Theta, r)$ can be accessed internally via an isomorphism $\forall_{\mathbb{U}}(\Theta, r) \cong \Theta^r$ where Θ^r is defined as follows: 1. $\Theta^r = \bot$, the initial context or empty presheaf, if r is not a variable up to shape isomorphism, 2. $(\Gamma, u : \mathbb{U}, \Delta)^u = (\Gamma, \forall u.\Delta)$ where $\forall u.\Delta$ universally quantifies the type of every normal variable in Δ, while its behaviour on shape variables depends on the structural rules.

We introduce a boundary predicate $r \in \partial\mathbb{U}$ [ND24], which we can give an eliminator if $\mathcal{S}$ is Reedy. Then an improved telescope-quantifying Gel/Ψ-type [BCM15, CH21] can be defined from transpension [ND24]. Writing $\mathsf{app}_{\mathbb{U}}$ for the co-unit of $\exists\!\!\!\!\!-_{\mathbb{U}} \dashv \forall_{\mathbb{U}}$, then under a mild condition on $\mathcal{S}/\mathbb{U}$ we also get an improved telescope-quantifying extent/Φ-rule [BCM15, CH21, ND24]:

CWOD:EXTENT
$$\frac{\Theta \vdash r : \mathbb{U} \qquad \Theta, \_ : r \in \partial\mathbb{U} \vdash c_\partial : C \qquad \forall_{\mathbb{U}}(\Theta, r), u : \mathbb{U} \vdash c_\forall : C[\mathsf{app}_{\mathbb{U}}]\{\_ : u \in \partial\mathbb{U} \mapsto c_\partial[\mathsf{app}_{\mathbb{U}}]\}}{\Theta \vdash \mathsf{extent}\, c_\partial\, (u.c_\forall)\, r : C}$$

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