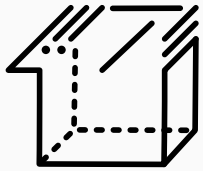

Towards Computational UIP in Cubical Agda

Yee-Jian Tan, Andreas Nuyts, Dominique Devriese

TYPES 2026 (Gothenburg, Sweden)
06 May 2026

DistriNet, KU Leuven, Belgium

Cubical-UIP

Cubical Agda [VMA21]

Based on [Coh+17] with De Morgan cubes.

- transp and hcomp
- I, *Glue* types, higher inductive types

Glue types (source of univalence) incompatible with Uniqueness of Identity Proofs (UIP) and K.

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Glue types (source of univalence) incompatible with Uniqueness of Identity Proofs (UIP) and K. (But we can disable it!)

FunExt, Quotients, possibly **UIP**... good for working with sets! [Coc19, Shu17]

Motivation

Choices for working with **sets in Agda** (with Uniqueness of Identity Proofs):

	FunExt	**UIP holds**	**UIP computes**	**Quotient**	**Projects**
Agda `--with-K`	❌	✅	✅	❌	(Many)
Agda + Setoids	❌	Setoid ⚠️	✅	Setoid ⚠️	[CND25, HC21]

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`--cubical=no-glue` + postulated UIP	✅	✅	❌	✅	[CNT26]

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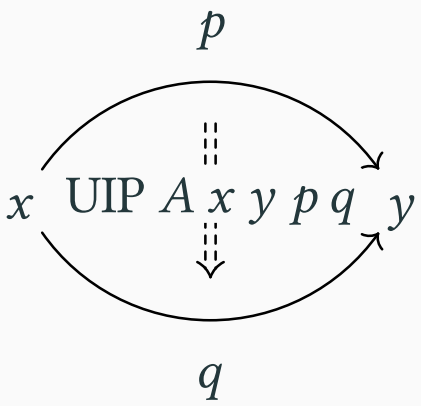
Cubical-UIP

✅ Agda `--cubical=no-glue` variant (merged in 2.9.0)

⌚ Agda `--cubical=uip` variant with a “UIP primitive” that recurses on types

UIP: working with h-Sets

UIP: squares with reflexive sides have a filler.



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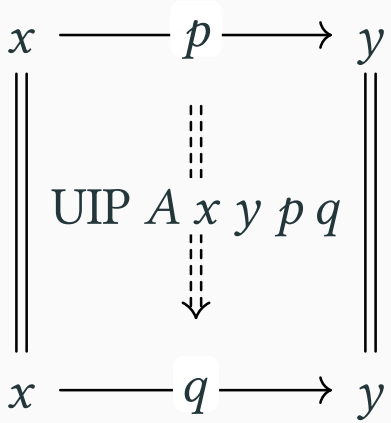

Reflexive sides

[1] [Agd25]

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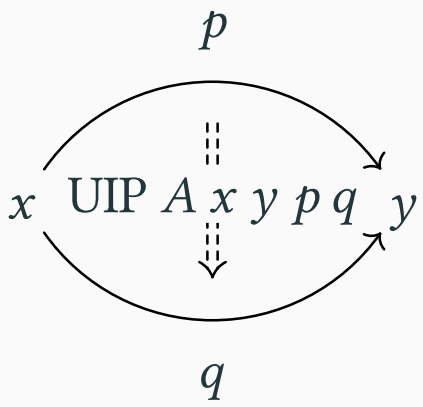

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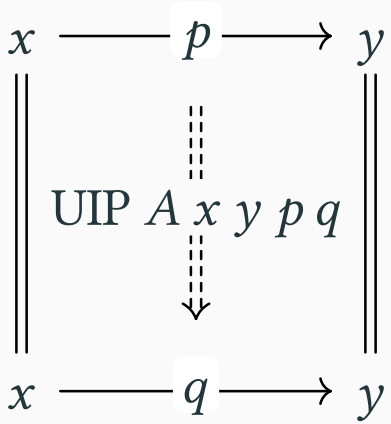

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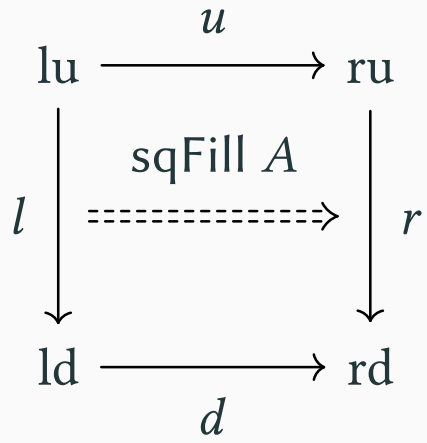

Square-filling

SqFill: any given square has a filler.

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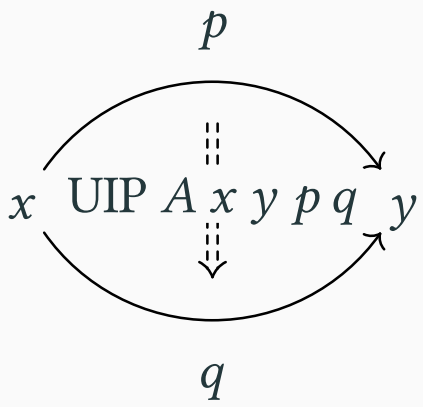

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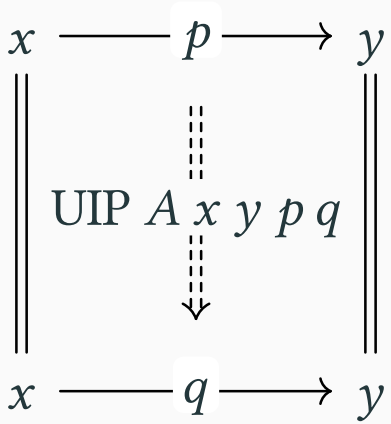

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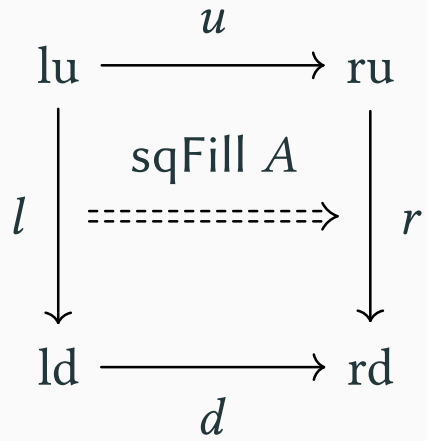

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$$\text{UIP} \leftrightarrow^{1} \text{SqFill}$$

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Base types: a proof term is directly given.

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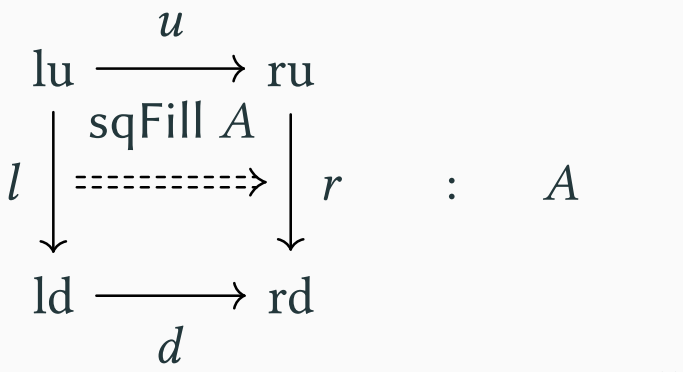
Implemented: **Pi**, **Sigma**, **PathP**; **List**, **Maybe**, **(Co-)products** ; Base types: **Unit**, **Bool**, **Nat**

Welcome to experiment: https://github.com/yeejian-tan/agda/tree/cubical-uip

Non-dependent vs Dependent



Homogenous SqFill

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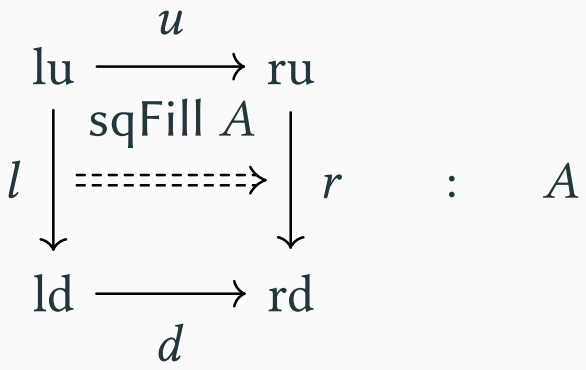

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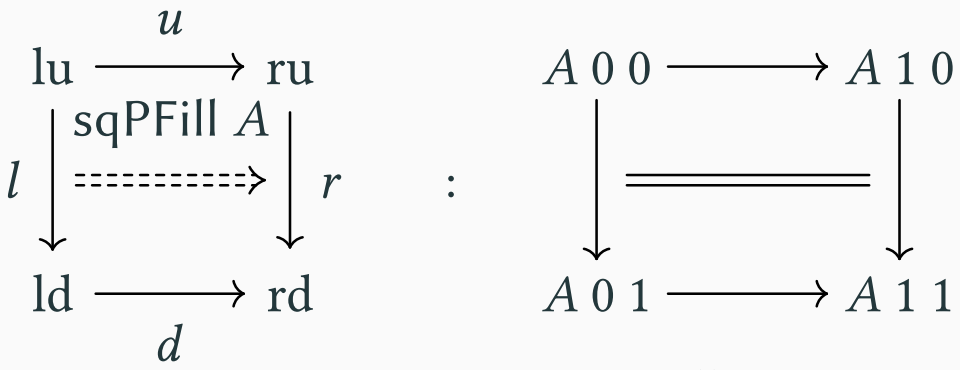

Heterogeneous SqPFill

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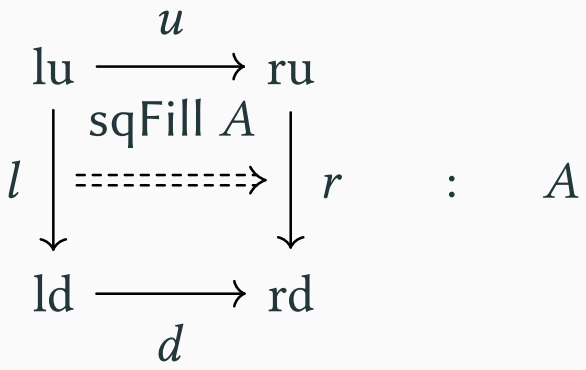

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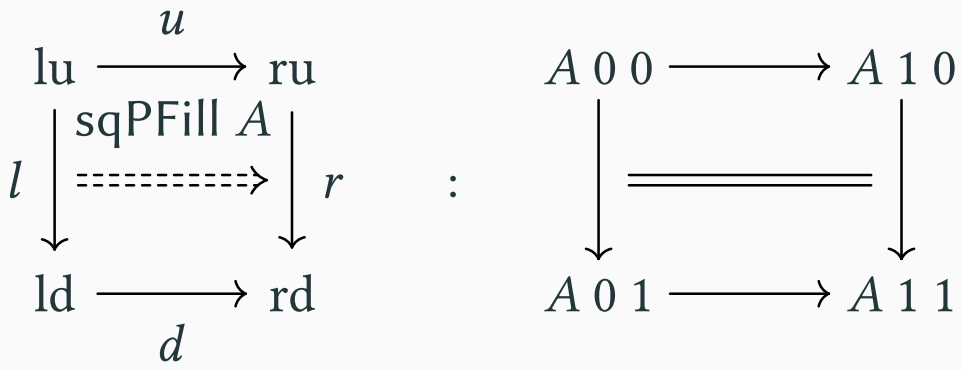

Heterogeneous SqPFill

SqPFill (heterogenous) $\leftrightarrow$ pointwise SqFill

Observation: Complexity of Preservation Proofs

Our proofs attempt to fill squares directly instead of instantiating more general results [TND25].

Preservation proofs differ in complexity for SqFill and SqPFill:

	SqFill (non-dependent)	SqPFill (dependent)
Pi	**Trivial** (no Kan operations)	**Complicated**: transport-fill-align
Sigma	**Complicated**: transport-fill-align	**Trivial** (no Kan operations)
Coproducts	**Encode-decode** (J, hcomp) [Uni13]	**Encode-decode** (J, comp) [Uni13]
Path Types	**Simple** (a single hcomp)	**Complicated**: transport-fill-align

Complexity of SqFill and SqPFill preservation proofs

Evaluation

Candidates:

- Chen, Nordvall Forsberg, and Tsai's TT-as-QIIRT [CNT26]
  Replacing UIP postulate by our sqFill primitive
- Bitvector Library in Katamaran [Huy+23]
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Do you have evaluation targets that **compute** over **UIP proofs**?

- Possibly need: **FunExt**, **Quotient types**?
- Would benefit from **computational UIP/proof-irrelevance**?
- No need: **univalence**?

Related Work

System	UIP	Equality … compute
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Cubical-UIP	Propositional (+ automation)	proofs
XTT	**Definitional**	proofs
Observational Type Theory (OTT)	**Definitional** (needs SProp)	**types**

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Observational Type Theory [Alt99, AMS07, PLT25]

- UIP needs proof-irrelevant propositions
- Equality **types** compute instead of **proofs**

$$\sim_{A \times B} = \sim_A \times \sim_B$$

(c.f. *isomorphic* in Cubical)

Open question: Universe 🪐

What happens to sqFill U?

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Option 1: U is an inductive-recursive universe

```
data U : Type₁ where
  Pi : (A : U) →  (B : El A → Type) → U
  ...
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Advantage: sqFill U (and hcomp U) compute by recursing on typecodes.

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Option 2: Universe is *not* closed

- sqFill U as a new type… ?
- sqFill only for small types?

Future work

Universes.

Records: nested Σ-types?

Datatypes:

- HITs: add sqFill constructor
- Non-HITs: add sqFill constructor; or "pattern-match" on corners (like hcomp)

Contribution

✅ `--cubical=no-glue`: (Cubical Agda - Glue)

⌚ `--cubical=uip`: (Cubical Agda - Glue + sqFill primitive recursing on types) [TND25]

Supports (Bool, Nat, Unit), (Pi, Sigma, PathP, List, Maybe, (Co-)products).

Welcome to experiment: https://github.com/yeejian-tan/agda/tree/cubical-uip

Questions?

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Bonus: Demo

See a square compute! 👀