Towards FaceTT: a generalization of intensional type systems with Glue

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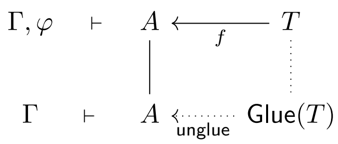
Abstract

Several dependent type systems feature an interval pseudotype $\mathbb{I}$, and a pseudotype of *face formulas*. Terms and types can depend on interval variables and on the truth of face formulas. Roughly, these features internalize aspects of the presheaf model underlying the theory. Examples include Cohen et al.'s (CCHM) cubical type theory, Angiuli et al.'s cartesian cubical type theory and Nuyts et al.'s ParamDTT, all of which use at least a Glue type whose rules reference face formulas.

We seek to generalize these systems to a single type system FaceTT, parametrized by a *shape theory*, which declares one or more *shape pseudotypes* such as $\mathbb{I}$, as well as *shape operations* and *face formula operations*. The automatically resulting inference rules for the dependent type system with the Glue type must then be sound w.r.t. the presheaf models on properly chosen base categories. Our goal is to show that FaceTT satisfies these semantic properties when it is supplied a suitable semantics for the shape theory. Similarly, we want FaceTT to satisfy computational properties, like reducibility of face formulas, when supplied with the right computational proofs about the shape theory.

Background There are many interesting dependent type theories that make special use of an interval, like Cohen et al.'s [CCHM17] cubical type theory (CCHM), Angiuli et al.'s [ABC+21] cartesian cubical type theory and Nuyts et al.'s [NVD17] parametric dependent type theory (ParamDTT). (More precisely, we consider a non-modal version of the latter parametric theory.) Upon closer inspection, these theories appear to present a common structure. All of them postulate the existence of a pseudotype, say $\mathbb{I}$, representing the formal interval with endpoints 0 and 1. Statements about variables in $\mathbb{I}$ then form a structure of *face formulas*, different for each type system. A face formula φ can be said to hold ($\Gamma \vdash \varphi$) and can appear anywhere in a context, e.g. $\Gamma, \varphi, \Delta \vdash J$, expressing the assumption that it holds.

These type systems also postulate a Glue type. Given a type A, as well as a type T and a function $f : T \to A$ which only exist when face formula φ holds, it allows one to expand T to a type $\mathsf{Glue}(T)$ which exists regardless of φ and is equal to the original T when φ does hold:



In cubical type theory, Kan fibrancy of Glue demands that f be an equivalence and Glue is used to *prove* univalence rather than postulate it like in book HoTT [UFP13]. ParamDTT uses Glue to prove that $A \to B$ implies $A \frown B$ and $B \frown A$, which is the relevant notion of parametric relatedness.

FaceTT: a unifying system with Glue Our goal is to define a type system FaceTT that takes a *shape theory* as a parameter and instantiates to the aforementioned type systems for corresponding choices of the shape theory. A shape theory consists of a list of shape pseudo-types (the above example type systems have only $\mathbb{I}$), shape operations on/between them (e.g., CCHM's cubical type theory allows for shape expressions like $1 - (i_1 \vee (0 \wedge i_2))$ in the free De Morgan algebra on the variables in scope and 0, 1), and face formula formers (e.g. Riehl-Shulman's directed type theory [RS17] adds an atom $(i_1 \leq i_2)$ besides $(i_1 = i_2)$ and the logical connectives[1]).

We want FaceTT to have general presheaf semantics, assuming a semantics of the shape theory. Such a shape theory semantics comprises a choice of base category $\mathcal{W}$ for the presheaf semantics, along with base objects to semantically represent the shapes and base morphisms for the shape operators, as well as suitable semantic representations of the face formula formers.

Similarly, the computational behavior of FaceTT is determined by a description of the computation of the supplied shape theory. The working of the Glue type requires that face formulas that are true, even due to assumptions in the context, reduce to $\top$. An intended theorem, therefore, is that equality and entailment of face formulas are decidable. This is a required step towards decidability of equality of terms and types in FaceTT. A clear necessary condition is the decidability of entailment of atomic face formulas from atomic face formulas.

Status and Conclusion We note that Rose-Licata recently proved entailment of face formulas decidable for a fragment of the internal language of cubical sets [RL25]. It is, however, a first-order language with no shape operations, and thus markedly different from our approach.

A step in the direction of FaceTT was Nuyts's unifying overview [Nuy20] of typing rules and their soundness w.r.t. presheaf categories [Hof97, HS97].

However, in Nuyts's account, the semantics and computational aspects of shape theories are not always clear, and neither is the boundary between shape theories and FaceTT. For example, Nuyts sketched a context rewrite system for face formulas. When assuming, e.g., $\varphi \wedge \chi$ in a context, it concludes that $\varphi \vee \psi = \top \vee \psi = \top$. It is not immediately clear how to specify the system's behavior on all face formula formers, though, which currently obstructs decidability proofs. As it stands, some rules appear problematic, like the overly general EQ-RED [Nuy20, Fig. 6.4] which is unclearly interwoven with the shape pseudotypes.

Starting from Nuyts's rules, we are working to define the FaceTT type system, its reduction rules, semantics and some meta-theoretic results. We intend FaceTT to standardize common patterns in type theories with presheaf semantics, offering reusable syntax, typing and computation rules, semantics and basic meta-theory that can be used as a starting point for type theories with presheaf models.

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[1] We do note that, unlike us, Riehl-Shulman do not venture into decidability of equality of face formulas.

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