Lifting Profunctors to Presheaf Categories
v0.1.1

Andreas Nuyts
DistriNet, KU Leuven, Belgium

February 15, 2024

Abstract

It is well-known that a functor between base categories, leads to an adjoint triple between presheaf categories. Here, we discuss how a profunctor between base categories, leads to an adjoint pair between presheaf categories. The former result can be recovered from the latter: a functor has a companion and a conjoint profunctor, each of which leads to an adjoint pair between the presheaf categories. It turns out that these pairs have one functor in common and as such constitute an adjoint triple. We give explicit constructions, but the result also follows from well-known categorical facts, so no particular novelty is claimed.

Notation 0.1. We use the presheaf notations from earlier work [Nuy20, §2.3.1], concretely:

- The application of a presheaf $\Gamma \in \mathsf{Psh}(\mathcal{W})$ to an object $W \in \mathcal{W}$ is denoted $W \Rightarrow \Gamma$. An element $\gamma : W \Rightarrow \Gamma$ is called a **cell** of Γ of **shape** W.
- The **restriction** of $\gamma : W \Rightarrow \Gamma$ by $\varphi : V \to W$, i.e. the action of Γ on φ applied to γ, is denoted $\gamma \circ \varphi$ or $\gamma\varphi : V \Rightarrow \Gamma$.
- The application of a presheaf morphism $\sigma : \Gamma \to \Delta$ to $\gamma : W \Rightarrow \Gamma$ is denoted $\sigma \circ \gamma$ or $\sigma\gamma$. By naturality of σ, we have $\sigma \circ (\gamma \circ \varphi) = (\sigma \circ \gamma) \circ \varphi$.

If $\mathcal{W}$ has a terminal object, then we call a cell $\top \Rightarrow \Gamma$ a **point**.

Notation 0.2. We will always regard **profunctors** as functors to presheaf categories, i.e. a profunctor $\mathcal{W}^{\mathsf{op}} \times \mathcal{V} \to \mathsf{Set}$ will be studied in swapped-curried form $\mathcal{V} \to \mathsf{Psh}(\mathcal{W})$.

1 Lifting Functors

This section is taken verbatim from [Nuy20, §2.3.8].

Theorem 1.1. Any functor $F : \mathcal{V} \to \mathcal{W}$ gives rise to functors $F_! \dashv F^* \dashv F_*$, with a natural isomorphism $F_! \circ \mathbf{y} \cong \mathbf{y} \circ F : \mathcal{V} \to \mathsf{Psh}(\mathcal{W})$. [Sta19, nLa24b]

The operation $\sqcup^* : \mathsf{Cat}^{\mathsf{coop}} \to \mathsf{Cat}$ sending $F : \mathcal{V} \to \mathcal{W}$ to $F^* : \mathsf{Psh}(\mathcal{W}) \to \mathsf{Psh}(\mathcal{V})$ is a strict 2-functor. Hence $\sqcup_!, \sqcup_* : \mathsf{Cat} \to \mathsf{Cat}$ are pseudofunctors.

Proof. Using quantifier symbols for ends and co-ends, we can define:

$$\begin{aligned}
W \Rightarrow F_!\Gamma &:= \exists V.(W \to FV) \times (V \Rightarrow \Gamma), \\
V \Rightarrow F^*\Delta &:= FV \Rightarrow \Delta \\
W \Rightarrow F_*\Gamma &:= \forall V.(FV \to W) \to (V \Rightarrow \Gamma) \quad = \quad (F^*\mathbf{y}W \to \Gamma).
\end{aligned}$$

By the co-Yoneda lemma, we have, naturally in W:

$$\begin{aligned}
W \Rightarrow F_!\mathbf{y}V &= \exists V'.(W \to FV') \times (V' \to V) \\
&\cong (W \to FV) \quad = \quad (W \Rightarrow \mathbf{y}FV),
\end{aligned}$$

i.e. $F_!\mathbf{y}V \cong \mathbf{y}FV$.

Adjointness also follows from applications of the Yoneda and co-Yoneda lemmas. [Sta19]

It is evident from the definition of $\sqcup^*$ that it preserves identity and composition on the nose, and easy to check that it turns around natural transformations. By uniqueness of the left/right adjoint, $\sqcup_!$ and $\sqcup_*$ are then pseudofunctors. $\square$

2 Lifting Profunctors

Theorem 2.1. Any profunctor $F : \mathcal{V} \to \mathsf{Psh}(\mathcal{W})$ gives rise to functors $F_\odot \dashv F^\odot$, with a natural isomorphism $F_\odot \circ \mathbf{y} \cong F$. The functor $F_\odot$ is therefore called the **Yoneda-extension** of F [nLa24c].

Proof. Using quantifier symbols for ends and co-ends, we define:

$$\begin{array}{rcl} W \Rightarrow F_\odot\Gamma & := & \exists V.(W \Rightarrow FV) \times (V \Rightarrow \Gamma), \\ V \Rightarrow F^\odot\Delta & := & \forall W.(W \Rightarrow FV) \to (W \Rightarrow \Delta). \end{array}$$

By the co-Yoneda lemma, we have, naturally in W:

$$\begin{array}{rcl} W \Rightarrow F_\odot\mathbf{y}V & = & \exists V'.(W \Rightarrow FV') \times (V' \to V) \\ & \cong & (W \Rightarrow FV), \end{array}$$

i.e. $F_\odot\mathbf{y}V \cong FV$.

To see adjointness, we have a chain of natural isomorphisms:

$$\begin{array}{rcl} F_\odot\Gamma \to \Delta & = & \forall W.(W \Rightarrow F_\odot\Gamma) \to (W \Rightarrow \Delta) \\ & = & \forall W.(\exists V.(W \Rightarrow FV) \times (V \Rightarrow \Gamma)) \to (W \Rightarrow \Delta) \\ & \cong & \forall W.\forall V.(W \Rightarrow FV) \to (V \Rightarrow \Gamma) \to (W \Rightarrow \Delta) \\ & \cong & \forall V.(V \Rightarrow \Gamma) \to \forall W.(W \Rightarrow FV) \to (W \Rightarrow \Delta) \\ & = & \forall V.(V \Rightarrow \Gamma) \to (V \Rightarrow F^\odot\Delta) \\ & = & \Gamma \to F^\odot\Delta. \end{array}$$

$\square$

Remark 2.2. When we view profunctors $\mathcal{W} \nrightarrow \mathcal{V}$ as functors $\mathcal{W}^{\mathsf{op}} \times \mathcal{V} \to \mathsf{Set}$, then

- the identity profunctor $\mathcal{V} \nrightarrow \mathcal{V}$ is the functor $\mathrm{Hom} : \mathcal{V}^{\mathsf{op}} \times \mathcal{V} \to \mathsf{Set}$,
- the composite of $\mathcal{Q} : \mathcal{W} \nrightarrow \mathcal{V}$ and $\mathcal{P} : \mathcal{V} \nrightarrow \mathcal{U}$ is given by:

$$(\mathcal{Q} \otimes \mathcal{P})(U, W) = \exists V.\mathcal{Q}(W, V) \times \mathcal{P}(V, U).$$

If instead we view profunctors $\mathcal{W} \nrightarrow \mathcal{V}$ as functors $\mathcal{V} \to \mathsf{Psh}(\mathcal{W})$, then

- the identity profunctor $\mathcal{V} \nrightarrow \mathcal{V}$ is the functor $\mathbf{y} : \mathcal{V} \to \mathsf{Psh}(\mathcal{V})$,
- the composite of $G : \mathcal{V} \to \mathsf{Psh}(\mathcal{W})$ and $F : \mathcal{U} \nrightarrow \mathsf{Psh}(\mathcal{V})$ is given by:

$$\begin{array}{rcl} W \Rightarrow (F \ggg G)U & = & \exists V.(W \Rightarrow GV) \times (V \Rightarrow FU) \\ & = & W \Rightarrow G_\odot FU. \end{array}$$

On a side note: yes, Psh is a 2-monad[1] with unit $\mathbf{y}$ and yes, the 2-category of categories and profunctors is the Kleisli-2-category of Psh, just like the category of sets and relations is the Kleisli-category of the powerset monad.

Theorem 2.3. The operations $\sqcup_\odot$ and $\sqcup^\odot$ are pseudofunctors w.r.t. identity and composition of profunctors.

[1] If we ignore size issues, otherwise it is may be only a relative 2-monad.

Proof. By adjointness, it suffices to show this for $\sqcup_\odot$. For the identity, we have:

$$\begin{aligned} V \Rrightarrow \mathbf{y}_\odot \Gamma &= \exists V'.(V \Rrightarrow \mathbf{y}V') \times (V' \Rrightarrow \Gamma) \\ &= \exists V'.(V \to V') \times (V' \Rrightarrow \Gamma) \\ &\cong V \Rrightarrow \Gamma \end{aligned}$$

For composition, we have:

$$\begin{aligned} W \Rrightarrow (G_\odot F)_\odot \Theta &= \exists U.(W \Rrightarrow G_\odot FU) \times (U \Rrightarrow \Theta) \\ &= \exists U.(\exists V.(W \Rrightarrow GV) \times (V \Rrightarrow FU)) \times (U \Rrightarrow \Theta) \\ &\cong \exists V.(W \Rrightarrow GV) \times (\exists U.(V \Rrightarrow FU) \times (U \Rrightarrow \Theta)) \\ &= \exists V.(W \Rrightarrow GV) \times (V \Rrightarrow F_\odot \Theta) \\ &= W \Rrightarrow G_\odot F_\odot \Theta. \end{aligned}$$

We do not consider the associahedra. □

3 Lifting Companion and Conjoint Profunctors

Remark 3.1. To a functor $F : \mathcal{V} \to \mathcal{W}$, we can generally associate

- a companion profunctor $\mathrm{Hom}(F\sqcup, \sqcup) : \mathcal{V}^{\mathsf{op}} \times \mathcal{W} \to \mathsf{Set}$,
- a conjoint profunctor $\mathrm{Hom}(\sqcup, F\sqcup) : \mathcal{W}^{\mathsf{op}} \times \mathcal{V} \to \mathsf{Set}$.

Viewing profunctors as functors to the presheaf category, we get instead:

- as the companion, the functor $F^* \circ \mathbf{y} : \mathcal{W} \to \mathsf{Psh}(\mathcal{V})$,
- as the conjoint, the functor $\mathbf{y} \circ F : \mathcal{V} \to \mathsf{Psh}(\mathcal{W})$.

Theorem 3.2. For a functor $F : \mathcal{V} \to \mathcal{W}$, we have:

$$(\mathbf{y} \circ F)_\odot = F_!, \qquad (\mathbf{y} \circ F)^\odot \cong F^* \cong (F^* \circ \mathbf{y})_\odot, \qquad F_* = (F^* \circ \mathbf{y})^\odot.$$

Proof. We have

$$\begin{aligned} W \Rrightarrow (\mathbf{y} \circ F)_\odot \Gamma &= \exists V.(W \Rrightarrow \mathbf{y}FV) \times (V \Rrightarrow \Gamma) \\ &= \exists V.(W \to FV) \times (V \Rrightarrow \Gamma) \\ &= F_!\Gamma, \\ V \Rrightarrow (\mathbf{y} \circ F)^\odot \Delta &= \forall W.(W \Rrightarrow \mathbf{y}FV) \to (W \Rrightarrow \Delta) \\ &= \forall W.(W \to FV) \to (W \Rrightarrow \Delta) \\ &\cong FV \Rrightarrow \Delta \quad = \quad V \Rrightarrow F^*\Delta, \\ V \Rrightarrow (F^* \circ \mathbf{y})_\odot \Delta &= \exists W.(V \Rrightarrow F^*\mathbf{y}W) \times (W \Rrightarrow \Delta) \\ &= \exists W.(FV \to W) \times (W \Rrightarrow \Delta) \\ &\cong FV \Rrightarrow \Delta \quad = \quad V \Rrightarrow F^*\Delta, \\ W \Rrightarrow (F^* \circ \mathbf{y})^\odot \Gamma &= \forall V.(V \Rrightarrow F^*\mathbf{y}W) \to (V \Rrightarrow \Gamma) \\ &= F^*\mathbf{y}W \to \Gamma \quad = \quad W \Rrightarrow F_*\Gamma. \end{aligned}$$ □

4 At a Higher Level

I personally like to brute-force presheaf details, but the existence of pseudofunctorial operations $\sqcup_\odot$ and $\sqcup^\odot$ can also be argued at a higher level. It is known that $\mathsf{Psh}(\mathcal{W})$ is the free cocompletion of $\mathcal{W}$, with $\mathbf{y} : \mathcal{W} \to \mathsf{Psh}(\mathcal{W})$ embedding the original. Then it is clear that a profunctor $F : \mathcal{V} \to \mathsf{Psh}(\mathcal{W})$

can be extended cocontinuously along $\mathbf{y} : \mathcal{V} \to \mathsf{Psh}(\mathcal{V})$ to $F_{\odot} : \mathsf{Psh}(\mathcal{V}) \to \mathsf{Psh}(\mathcal{W})$, since $\mathsf{Psh}(\mathcal{W})$ is cocomplete. Now $F_{\odot}$ is a cocontinuous functor between cocomplete categories and thus [nLa24a] has a right adjoint $F^{\odot}$.

If we want to define $\sqcup_!$ at a high level, we would simply specialize the above reasoning to a *pure* functor, i.e. we *define* $F_! = (\mathbf{y} \circ F)_{\odot}$. We could similarly *define* $F_* = (F^* \circ \mathbf{y})^{\odot}$, where F^* is defined by precomposition, as usual.

Finally, we need to show

$$(\mathbf{y} \circ F)^{\odot} \cong F^* \cong (F^* \circ \mathbf{y})_{\odot}.$$

Since colimits in presheaf categories are taken pointwise, F^* is clearly cocontinuous, so we can prove that $F^* \cong (F^* \circ \mathbf{y})_{\odot}$ by proving that they extend the same functor $\mathcal{W} \to \mathsf{Psh}(\mathcal{V})$ along $\mathbf{y} : \mathcal{W} \to \mathsf{Psh}(\mathcal{W})$. Now by construction of $\sqcup_{\odot}$, we have $(F^* \circ \mathbf{y})_{\odot} \circ \mathbf{y} \cong F^* \circ \mathbf{y}$, so this is in fact trivial.

Finally, using just a small amount of brute force, the construction of right adjoint functors to cocontinuous functors between presheaf categories, works by setting $(W \Rightarrow R\Gamma) := (L\mathbf{y}W \to \Gamma)$. When we apply this to $(\mathbf{y} \circ F)^{\odot}$, we find:

$$\begin{aligned} W \Rightarrow (\mathbf{y} \circ F)^{\odot}\Gamma &= (\mathbf{y} \circ F)_{\odot}\mathbf{y}W \to \Gamma \\ &\cong \mathbf{y}FW \to \Gamma \\ &\cong W \Rightarrow F^*\Gamma. \end{aligned}$$

References

[nLa24a] nLab authors. adjoint functor theorem, February 2024. Revision 78. URL: `https://ncatlab.org/nlab/show/adjoint+functor+theorem`.

[nLa24b] nLab authors. functoriality of categories of presheaves, February 2024. Revision 6. URL: `https://ncatlab.org/nlab/show/functoriality+of+categories+of+presheaves`.

[nLa24c] nLab authors. Yoneda extension, February 2024. Revision 14. URL: `https://ncatlab.org/nlab/show/Yoneda+extension`.

[Nuy20] Andreas Nuyts. *Contributions to Multimode and Presheaf Type Theory*. PhD thesis, KU Leuven, Belgium, 8 2020. URL: `https://lirias.kuleuven.be/3065223`.

[Sta19] The Stacks Project Authors. Stacks project. `http://stacks.math.columbia.edu`, 2019. Tags 00VC and 00XF.