A Vision for Natural Type Theory

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Abstract

In this brief and dense note, I lay out my view on how to proceed towards a higher-dimensional directed dependent type theory with interacting modalities for covariance and naturality/parametricity (natural type theory for short), ideally resulting in internal functorial actions for free and naturality theorems for free.

The first part highlights some ideas that I believe are obstructing further progress in this direction, and explains how to get around them: the Grothendieck construction which is the most well-known Σ-construction for categories but is often not the best choice, and the focus on forcing classifiers to contain themselves which is no longer a necessity in multimode type theory.

The second part gives some details on how to proceed towards a model of natural type theory.

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Part I
Grothendieck Construction Considered Harmful

This title is clickbait, of course, the Grothendieck construction has proven its worth. What I want to argue in this first part, is that it is too well-known for its own good, causing it to be the first resort when in need of a Σ-type for categories, obstructing the view on alternatives. I will point out some flaws of the Grothendieck construction and list a number of alternatives that are in some respects more panthodic:

Remark 0.1 (Panthodic)**.** In order to avoid confusion with the word 'natural' as in 'natural transformation', instead of natural as in non-contrived, we will use the neologism 'panthodic' (Ancient Greek: allroadsy, i.e. all roads would lead to that concept). The word is already in use in medicine with a completely different meaning.

Remark 0.2. Matters of size are mostly orthogonal to anything discussed in this note, so we will ignore them altogether. In particular, we assume there is a set of sets.

1 Well-known Constructions

1.1 Set Coproduct

Definition 1.1. Given a set A and a function $B : A \to \mathsf{Set}_{\in \mathsf{Set}}$ to the set of sets, we define their coproduct or Σ-set

$$\Sigma AB = \{(a, b) \mid a \in A, b \in B(a)\}. \tag{1}$$

There is a function $\pi_1 : \Sigma AB \to A : (a, b) \mapsto a$.

Remark 1.2 (Fibres)**.** The set coproduct turns a function $B : A \to \mathsf{Set}$ into a function $\pi_1 : \Sigma AB \to A$. This operation has an inverse, turning a function $f : T \to A$ into a function $f^{-1} : A \to \mathsf{Set} : a \mapsto f^{-1}(a)$ sending objects to their preimage under f. These operations are not inverse on the nose: they constitute an equivalence of categories.

1.2 Category of Elements

Definition 1.3. Given a category C and a covariant functor $\mathcal{D} : C \to \mathsf{Set}_{\in \mathsf{Cat}}$ to the category of sets, we define their category of elements

$$\int_C \mathcal{D} \tag{2}$$

as the category

- With objects (c, d) where $c \in C$ and $d \in \mathcal{D}(c)$,
- With morphisms $\varphi : (c_1, d_1) \to (c_2, d_2)$ where $\varphi : c_1 \to d_1$ and $\mathcal{D}(\varphi)(d_1) = d_2$.

There is a covariant functor $\pi_1 : \int_C \mathcal{D} \to C : (c, d) \mapsto c$.

Remark 1.4 (Characterization)**.** The equation $\mathcal{D}(\varphi)(d_1) = d_2$ can also be written as $d_1 \mapsto_{\mathcal{D}(\varphi)} d_2$, i.e. a morphism in the category of elements consists of a morphism φ in C and a mapping over φ in $\mathcal{D}$.

Remark 1.5 (Dualization)**.** By considering dualization, one additional concept arises: the category of elements of a *contravariant* functor $\mathcal{D} : C \to \mathsf{Set}$ can be formed as

$$\int_C \mathcal{D} := \left(\int_{C^{\mathrm{op}}} \mathcal{D}\right)^{\mathrm{op}}. \tag{3}$$

There is still a covariant projection $\pi_1 : \int_C \mathcal{D} \to C$.

Remark 1.6 (Fibres)**.** The category of elements turns a functor $\mathcal{D} : C \to \mathrm{Set}$ into a functor $\pi_1 : \int_C \mathcal{D} \to C$. A naïve attempt to construct an inverse as in remark 1.2 would turn a functor $F : \mathcal{T} \to C$ into a functor $F^{\mathrm{fib}} : C \to \mathrm{Set}$:

- sending an object $c \in C$ to its preimage under F. This preimage, however, is a category that in general may have non-trivial morphisms, i.e. it need not be a set.
- sending a morphism $\varphi : c \to d$ to – well, to what?

The issue here is that a morphism τ in T consists of a morphism φ in C and a morphism in the fibre of T *over* τ. This is in contrast to our observation in remark 1.4 where we saw that a morphism in the category of elements contains not a morphism but a *mapping* in the fibre. Clearly, a mapping over the identity is itself the identity, asserting that the preimage of c is discrete, solving the problem with the object part. The fact that it is a mapping relation, also asserts unique mappings over φ for the morphism part.

A functor $F : \mathcal{T} \to C$ satisfying the requirement that, for every $t_1 \in \mathcal{T}$ and $\varphi : Ft_1 = c_1 \to c_2$, there is a unique morphism $\tau : t_1 \to t_2$ such that $F\tau = \varphi$, is called a *discrete fibration* [nLa20b]. As such, the category of functors $C \to \mathrm{Set}$ is equivalent to the category of discrete fibrations over C.

1.3 Grothendieck Construction

Definition 1.7. Given a category C and a covariant functor $\mathcal{D} : C \to \mathrm{Cat}_{\in\mathrm{Cat}}$ to the category of categories, we define their Grothendieck construction (GC)

$$\int_C \mathcal{D} \tag{4}$$

as the category

- With objects (c, d) where $c \in C$ and $d \in \mathcal{D}(c)$,
- With morphisms $(\varphi, \chi) : (c_1, d_1) \to (c_2, d_2)$ where $\varphi : c_1 \to d_1$ and $\chi : \mathcal{D}(\varphi)(d_1) \to d_2$.

There is a covariant functor $\pi_1 : \int_C \mathcal{D} \to C : (c, d) \mapsto c$.

Remark 1.8 (Characterization)**.** Clearly, a morphism in the Grothendieck construction consists of a morphism in C and a morphism in $\mathcal{D}$ *over* it. Both morphisms point in the *same direction*.

Flaw 1.9 (Dualization)**.** Dualization can be done in two ways, leading to $2^2 - 1 = 3$ additional concepts.

First, we may simply postcompose $\mathcal{D}$ with $\mathrm{Op} : \mathrm{Cat}_{\in\mathrm{Cat}} \to \mathrm{Cat}_{\in\mathrm{Cat}}$, which is a covariant functor. This yields a construction whose morphisms consist of a morphism in C and a morphism in $\mathcal{D}$ over it, pointing in the *other* direction: a morphism (φ, χ) now consists of $\varphi : c_1 \to c_2$ and $\chi : d_2 \to F(\varphi)(d_1)$. We shall call this the **twisted Grothendieck construction**. Note that it is an *instance* of the GC.

Secondly, we can consider a contravariant functor $\mathcal{D}$ and define its Grothendieck construction as

$$\int_C \mathcal{D} := \left(\int_{C^{\mathrm{op}}} \mathrm{Op} \circ \mathcal{D} \right)^{\mathrm{op}} \tag{5}$$

yielding a category where a morphism (φ, χ) consists of $\varphi : c_1 \to c_2$ and $\chi : d_1 \to \mathcal{D}(\varphi)(d_2)$. We simply call this the **Grothendieck construction of the contravariant functor $\mathcal{D}$**.

Finally, we can combine.

While eq. (5) really feels like the panthodic way to generalize GCs to contravariant functors, the twisted construction really creates something new with the same input as the ordinary GC. When we consider the inputs C and $\mathcal{D}$, the orientation of morphisms in C and in $\mathcal{D}$ are not entangled whatsoever. Thus, the GC and the twisted GC are equally panthodic, yet the twisted GC is rarely used, if ever. This should indicate that something is off.

Flaw 1.10 (Fibres)**.** Again, it is infeasible to expect arbitrary slices $F : \mathcal{T} \to C$ to be isomorphic to a Grothendieck construction. Instead, we should restrict to Grothendieck fibrations (GFs) [nLa20d] over C. However, it is still impossible to reconstruct $\mathcal{D}$ from the GF $(\int_C \mathcal{D}, \pi_1)$ given by the GC. Indeed, in the GF, we can recognize *cartesian* arrows, which are arrows that can serve as mappings of the functorial action of $\mathcal{D}$. However, these mappings are not unique on the nose. Hence, while we retain the graph profunctor of $\mathcal{D}(\varphi)$ for any morphism φ in C, we do not retain the algebraic action of the functor $\mathcal{D}(\varphi)$. The issue is that the GC merges the mapping relation of $\mathcal{D}$ with the morphism relation of $\mathcal{D}$. This is not unrelated to flaw 1.9: indeed, by functoriality of $\mathcal{D}$, mappings in $\mathcal{D}$ over φ are aligned with φ, whereas the morphism relation of $\mathcal{D}$ can freely be flipped prior to taking the GC. Merging these relations is exactly what entangles them, and it turns out to be impossible to reconstruct the *precise* mapping relation of $\mathcal{D}$ from the GC.

2 Looking at Weak Object Classifiers

Definition 2.1. Let C be a category. A weak object classifier is a morphism $\omega : \widehat{\mathsf{U}} \to \mathsf{U}$ such that every other morphism $\varphi : x \to y$ is in some way a pullback of ω along some $\varphi^{\mathsf{fib}} : y \to \mathsf{U}$.

Seeing objects as collections, we can think of U as the collection of collections, and of $\widehat{\mathsf{U}}$ as the collection of pointed collections. Then φ^{fib} sends items to their fibre or preimage, and the morphism $x \to \widehat{\mathsf{U}}$ pairs up items of x with their fibre.

2.1 Sets

A weak object classifier of $\mathsf{Set}_{\in\mathsf{Cat}}$ is given by $\pi_1 : \Sigma(\mathsf{Set}_{\in\mathsf{Set}})\mathsf{Id} \to \mathsf{Set}_{\in\mathsf{Set}}$. The pullback along $B : A \to \mathsf{Set}_{\in\mathsf{Set}}$ is given by $\pi_1 : \Sigma AB \to A$. This makes Σ-sets a fairly panthodic notion.[1]

2.2 Categories

The category of categories $\mathsf{Cat}_{\in\mathsf{Cat}}$ does not have a weak object classifier per se. However, we can regard it as a full subcategory of the category of simplicial sets $\mathsf{SSet} = \mathsf{Psh}(\mathsf{Simplex})$ (namely the one whose objects satisfy the Segal condition), and then there is a morphism of simplicial sets $\omega : \widehat{\mathsf{Prof}} \to \mathsf{Prof}$ such that every functor $F : C \to \mathcal{D}$ is a pullback of ω. Concretely, the codomain $\mathsf{Prof}_{\in\mathsf{SSet}}$ of ω is the simplicial set:

- whose points are categories,
- whose lines are profunctors,
- whose triangles are profunctor morphisms $Q \circ \mathcal{P} \to \mathcal{R}$,
- whose higher simplices are commuting diagrams.

Note that a simplicial set morphism from a category C to $\mathsf{Prof}_{\in\mathsf{SSet}}$ is the same as a normal lax functor [nLa20e] to the 2-category of categories, profunctors and profunctor morphisms. The domain $\widehat{\mathsf{Prof}}$ of ω is the simplicial set:

- whose points (C, c) are pointed categories, i.e. categories C with a designated object $c \in C$,
- whose lines from (C, c) to $(\mathcal{D}, d)$ are pointed profunctors $(\mathcal{P}, \psi)$ with $\psi \in \mathcal{P}(c, d)$,
- whose triangles between $(\mathcal{P}, \psi)$, (Q, χ) and $(\mathcal{R}, \rho)$ are profunctor morphisms $Q \circ \mathcal{P} \to \mathcal{R}$ sending $\chi \circ \psi$ to ρ,

[1] The reader may object that a weak object classifier is not a universal thing. This is true, but note that if there are two weak object classifiers, these are at least a pullback of one another, implying conceptually that they have at least the same fibres. More importantly, what we *will* see is that for categories, we get something quite different from the Grothendieck construction.

- whose higher simplices are inherited fully faithfully from Prof, i.e. they are commutative diagrams with no additional condition regarding pointedness.

Now take a category C and a simplicial set morphism $\mathcal{D} : C \to \mathsf{Prof}_{\in \mathsf{SSet}}$. Then the pullback of ω is given by the following concept:

Definition 2.2 (Σ-category of a category and a normal lax functor)**.** Given a category C and a simplicial set morphism $\mathcal{D} : C \to \mathsf{Prof}_{\in \mathsf{SSet}}$, the Σ-category $\Sigma C\mathcal{D}$ is the category:

- with objects (c, d) with $c \in C$ and $d \in \mathcal{D}(c)$,
- with morphisms $(\varphi, \chi) : (c_1, d_1) \to (c_2, d_2)$ with $\varphi : c_1 \to c_2$ and $\chi : \mathcal{D}(\varphi)(d_1, d_2)$,
- with $\mathrm{id}_{(c,d)} = (\mathrm{id}_c, \mathrm{id}_d)$ (this is well-typed because $\mathcal{D}(\mathrm{id}_c)(d, d) = \mathrm{Hom}_{\mathcal{D}(c)}(d, d)$ as Hom is the identity profunctor),
- with $(\varphi', \chi') \circ (\varphi, \chi) = (\varphi' \circ \varphi, \Downarrow (\chi' \circ \chi))$ where $\Downarrow$ is the compositor of $\mathcal{D}$.

There is an obvious functor $\pi_1 : \Sigma C\mathcal{D} \to C$.

We can thus regard the above concept as fairly panthodic.

Remark 2.3 (Characterization)**.** A morphism in $\Sigma C\mathcal{D}$ consists of a morphism φ in C and a morphism χ *over* φ in $\mathcal{D}$, pointing in the same direction.

Remark 2.4 (Dualization)**.** Note that Op as a simplicial set morphism from Prof to Prof is now contravariant as it flips the direction of profunctors: $\mathrm{Op} : \mathsf{Prof}^{\mathrm{op}} \to \mathsf{Prof}$.

If we want a covariant first projection, it seems that the Σ-category can really be dualized only in one way: we can consider $\Sigma C\mathcal{D}$ for $\mathcal{D} : C \to \mathsf{Prof}^{\mathrm{op}}_{\in \mathsf{SSet}}$, by defining $\Sigma C\mathcal{D} := \Sigma C(\mathrm{Op} \circ \mathcal{D}) = (\Sigma(C^{\mathrm{op}})\mathcal{D})^{\mathrm{op}}$. Here, paired morphisms φ and χ point in opposite directions.

Remark 2.5 (Fibres)**.** The Σ-category has an inverse construction, which builds from a functor $F : \mathcal{T} \to C$ a simplicial set morphism $F^{\mathrm{fib}} : C \to \mathsf{Prof}_{\in \mathsf{SSet}}$:

- $F^{\mathrm{fib}}(c)$ is the category $F^{-1}(c)$,
- $F^{\mathrm{fib}}(\varphi)(t_1, t_2)$ is the set of morphisms $\tau : t_1 \to t_2$ such that $F\tau = \varphi$,
- The compositors are given by composition.

These operations are not inverse on the nose but up to (an appropriate notion of) isomorphism.

The Σ-category specializes to earlier concepts:

- There is a simplicial set morphism $\mathrm{Disc} : \mathsf{Set}_{\in \mathsf{Cat}} \to \mathsf{Prof}_{\in \mathsf{SSet}}$ sending sets to discrete categories and functions to their graph profunctor. Now $F : \mathcal{T} \to C$ is a discrete fibration if and only if F^{fib} factors over Disc.
- Assume choice[2]. There is a simplicial set morphism $\mathrm{Gro} : \mathsf{Cat}_{\in \mathsf{Cat}} \to \mathsf{Prof}_{\in \mathsf{SSet}}$ sending functors $F : \mathcal{A} \to \mathcal{B}$ to their graph profunctor $\mathrm{Hom}_{\mathcal{B}}(F\sqcup, \sqcup) : \mathcal{A} \nrightarrow \mathcal{B}$. Now F is a Grothendieck fibration if and only if F^{fib} factors over Gro.
- Still assume choice. We have a simplicial set morphism $\mathrm{Op} \circ \mathrm{Gro} \circ \mathrm{Op} : \mathsf{Cat}^{\mathrm{op}}_{\in \mathsf{Cat}} \to \mathsf{Prof}_{\in \mathsf{SSet}}$ sending functors $F : \mathcal{A} \to \mathcal{B}$ to $\mathrm{Hom}_{\mathcal{B}}(\sqcup, F\sqcup) : \mathcal{B} \nrightarrow \mathcal{A}$. Now F is a Grothendieck opfibration (corresponding to the GC of a contravariant functor) if and only if F^{fib} factors over $\mathrm{Op} \circ \mathrm{Gro} \circ \mathrm{Op}$.
- Recall that the twisted GC is an instance of the GC (and vice versa). Hence we have no special twisted fibrations.

Flaw 2.6. If we carefully analyze how definition 2.2 addresses the flaws of the GC, we see that it simply asks the inputs not to contain an algebraic action, so that the Σ-category does not have to encode it in a mapping relation somehow. This is of course the coward's solution. What we should aim for is to accept an algebraic action and to retain it. This is achieved in the next section.

[2] in order to spawn algebraic actions

3 Liberating Classifiers from Forced Self-Classification

I think the following is very important:

> We should not force classifiers to be an element of themselves.
> We should accept and love classifiers the way they are.

All the world's literature is not an example of a book. Mankind is not a human. And this is not a matter of size: mankind is also not a very big human. Forcing these things to be otherwise is unreasonable. Self-classification is – in general – a bad idea. It may be inviting in collections of collections such as Set and Cat, but even there, self-classification involves disregard for important structure.

We saw that the weak object classifier of $\mathsf{Set}_{\in \mathsf{Cat}}$ is $\mathsf{Set}_{\in \mathsf{Set}}$.[3] But when we regard Set as a weak object classifier, we regard it as a set, and that means we underappreciate it. Set has more structure than that of a set. It is a category, with functions as morphisms. Better even, it is a pro-arrow equipped category (or equipment for short), with relations as pro-arrows.

We saw that we had to leave Cat to consider a classifier of categories. But the object $\mathsf{Prof}_{\in \mathsf{SSet}}$ that we considered, again underappreciates the collection of categories. This collection can be endowed with profunctors as pro-arrows and functors as arrows. It, too, is an equipment.

The collection Eqmnt of equipments, in turn, is richer than an equipment.

3.1 Intermezzo: *n*-Fold Categories, *n*-Categories and Pro-arrow Equipments

***n*-Fold Categories** A double category [nLa20c] is a category internal to Cat, i.e. it consists of a category of objects, a category of morphisms, and source, target and composition functors. If we unwrap this definition, then we find that we have elements in four shapes:

- Objects (objects of the category of objects),
- Horizontal morphisms (morphisms of the category of objects),
- Vertical morphisms (objects of the category of morphisms),
- Squares (morphsims of the category of morphisms).

These things can be composed in two dimensions.

Although it doesn't sound that way, this definition is symmetric: if we swap the horizontal and vertical dimensions, then we still have a double category.

This concept is straightforwardly generalized to *n*-fold categories.

***n*-Categories** A (strict) 2-category is a category whose Hom-sets are not sets but categories. Composition is functorial. If we unwrap this definition, then we find that we have elements in three shapes:

- Objects,
- Morphisms (objects of the Hom-categories),
- 2-morphisms (morphisms of the Hom-categories).

This notion is generalized to *n*-categories, where Hom-sets are $(n-1)$-categories.

An example of a 2-category is the category Cat,

- whose objects are categories,
- whose morphisms are functors,
- whose 2-morphisms are natural transformations.

[3] Strictly speaking, $\pi_1 : \Sigma \mathrm{SetId} \to \mathrm{Set}$.

Cheng and Lauda [CL04] accessibly explain Tamsamani and Simpson's approach towards defining n-categories: A 2-category can be defined as a double category in which all horizontal morphisms are the identity, i.e. the horizontal dimension is only used for squares. A 3-category can be defined as a triple category in which the horizontal dimension is only used for squares and the third dimension only for cubes. Etc.

In the 2-category Cat, we would have

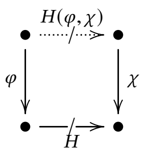
- Categories as objects,
- Functors as vertical morphisms,
- Trivial horizontal morphisms,
- Natural transformations as squares.

Pro-arrow Equipments Tamsamani and Simpson's approach is fun because it gets better if you remove their condition. We can view Cat as a *double* category:

- whose objects are categories,
- whose vertical morphisms are functors,
- *whose horizontal morphisms are profunctors*,
- whose squares are profunctor morphisms $\mathcal{P}(␣, ␣) \to Q(F␣, G␣)$. Note that if the profunctor sides are the identity (i.e. Hom), then squares are exactly natural transformations.

A category equipped with pro-arrows, a pro-arrow equipment or just an equipment [nLa20a] is a double category (whose vertical morphisms are called morphisms, and whose horizontal morphisms are called pro-arrows and denoted with the $\nrightarrow$ arrow) satisfying an additional condition that can be phrased in two equivalent ways:

1. Every diagram



(6)

   has a universal filler (meaning that every square with sides $\varphi \circ \varphi'$ and $\chi \circ \chi'$ and bottom H has to factor through it uniquely) whose top pro-arrow we call $H(\varphi, \chi)$.

2. Every morphism $\varphi : x \to y$ has a companion $\mathrm{Id}(\varphi, \mathrm{id}_y) : x \nrightarrow y$ and a conjoint $\mathrm{Id}(\mathrm{id}_y, \varphi) : y \nrightarrow x$. Companion and conjoint means that the identity squares on these pro-arrows factor horizontally over the morphism φ.

One can prove that these conditions are equivalent, and luckily so, because the first condition is universal but feels asymmetric w.r.t. the orientation of the morphisms, whereas the second condition is symmetric w.r.t. orientation but may feel non-universal.

An example of an equipment is Cat, with functors as morphisms and profunctors as pro-arrows. The top of the $\sqcup$-shaped box is given by $Q(F␣, G␣)$.

Another example is Set, with functions as morphisms and relations as pro-arrows. The top of the $\sqcup$-shaped box is given by $S(f␣, g␣)$.

3.2 The Upward Set Sum: The Category of Elements

As we have seen, the category of elements had excellent properties: a sensible characterization, only sensible dualizations, and it is invertible up to isomorphism.

One panthodic way to arrive at the definition of the category of elements is the following: We want to take a coproduct of sets. We regard Set as a category. Hence, we will also index over a category using a functor $\mathcal{D} : C \to \text{Set}$. The result is the category $\int_C \mathcal{D}$. It is worth noting that $\int_C \mathcal{D}$ is also obtained as the pullback of $\widehat{\text{Set}} \to \text{Set}$ along $\mathcal{D}$, where $\widehat{\text{Set}}$ is the category of pointed sets.

We can in fact regard Set as an equipment, leading to a richer notion:

Definition 3.1. Given an equipment C and an equipment functor $\mathcal{D} : C \to \text{Set}_{\in\mathsf{Eqmnt}}$ to the equipment of sets, we define their equipment of elements

$$\int_C \mathcal{D} \tag{7}$$

as the equipment

- With objects (c, d) where $c \in C$ and $d \in \mathcal{D}(c)$,
- With morphisms $\varphi : (c_1, d_1) \to (c_2, d_2)$ where $\varphi : c_1 \to d_1$ and $\mathcal{D}(\varphi)(d_1) = d_2$,
- With pro-arrows $\bar{c} : (c_1, d_1) \nrightarrow (c_2, d_2)$ where $\bar{c} : c_1 \nrightarrow c_2$ and $\mathcal{D}(\bar{c})(d_1, d_2)$.

There is a covariant functor $\pi_1 : \int_C \mathcal{D} \to C : (c, d) \mapsto c$.

Again, this can also be found as the pullback of $\widehat{\text{Set}} \to \text{Set}$ along $\mathcal{D}$, where $\widehat{\text{Set}}$ is the equipment of pointed sets, point-preserving functions and point-respecting relations.

From the pullback approach, we see that these notions of sums are still related to certain classifiers, though these no longer classify *all* morphisms.

We call these notions **upward sums** because the result lives in the category where Set lives, rather than within Set.

Note that the objects of the upward sum are essentially objects of C with extra structure. In modal type theory, we will say that (c, d) depends continuously on c and structurally on d. The structural modality lifts d from a set to a category, turning the mapping relation into a morphism relation, so that a morphism in the category of elements consists of a morphism in C and a mapping in $\mathcal{D}$.

3.3 The Downward Set Sum: The Colimit

With the same inputs and the desire to obtain a set, we should take the colimit, or a generalization to equipments. We will denote this as $\exists C\mathcal{D}$. Then there is no longer a first projection, but there is a co-cone of injections. This co-cone can be seen as a natural transformation $\forall c.\mathcal{D}(c) \to \exists C\mathcal{D}$. Thus, pairs (c, d) depend naturally on c. The natural/parametric modality lowers c from category to a set, turning morphisms into mappings (along the identity, i.e. equalities), effectively identifying (c_1, d_1) and (c_2, d_2) whenever there is a morphism between these in the category of elements.

3.4 The Upward Category Sum

In the same spirit as above, we can define an upward sum of categories. We regard Cat as an equipment, so we will take the sum of an equipment C and an equipment functor $\mathcal{D} : C \to \text{Cat}_{\in\mathsf{Eqmnt}}$:

Definition 3.2. Given an equipment C and an equipment functor $\mathcal{D} : C \to \text{Cat}_{\in\mathsf{Eqmnt}}$, we define their upward sum as the equipment $\int_C \mathcal{D}$:

- with objects (c, d) where $c \in C$ and $d \in \mathcal{D}(c)$,
- with morphisms $\varphi : (c_1, d_1) \to (c_2, d_2)$ where $\varphi : c_1 \to d_1$ and $\mathcal{D}(\varphi)(d_1) = d_2$ i.e. $d_1 \mapsto_{\mathcal{D}(\varphi)} d_2$,

- with pro-arrows $(\bar{c}, \bar{d}) : (c_1, d_1) \nrightarrow (c_2, d_2)$ where $\bar{c} : c_1 \nrightarrow c_2$ and $\bar{d} \in \mathcal{D}(\bar{c})(d_1, d_2)$ i.e. $d_1 \to_{\mathcal{D}(\bar{c})} d_2$,
- with squares $\bar{\varphi}$

$$\begin{array}{ccc} \bullet & \xrightarrow{(\bar{c}_1, \bar{d}_1)} & \bullet \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ \bullet & \xrightarrow[(\bar{c}_2, \bar{d}_2)]{} & \bullet \end{array} \tag{8}$$

  from C such that $\mathcal{D}(\bar{\varphi})(\bar{d}_1) = \bar{d}_2$.

There is an equipment functor $\pi_1 : \int_C \mathcal{D} \to C$.

This construction can again be obtained by taking the pullback of $\widehat{\mathsf{Cat}} \to \mathsf{Cat}_{\in \mathsf{Eqmnt}}$ along $\mathcal{D}$, where $\widehat{\mathsf{Cat}}$ is the equipment of pointed categories, point-preserving functors, pointed profunctors and point-preserving squares.

We see that the structural modality lifts d from the category $\mathcal{D}(c)$ to the equipment $\int_C \mathcal{D}$ by turning mappings into morphisms and morphisms into pro-arrows. This has the extremely important feature that we get to consider heterogeneous morphisms over pro-arrows. Thanks to the companion, the conjoint and composition of pro-arrow, any zigzag of morphisms in C gives rise to a pro-arrow, over which we can now consider morphisms. This answers the question of bridges that I raised in my master thesis exploring higher directed type theory [Nuy15].

Another important feature is that we now have the mapping relation of $\mathcal{D}$ fully encoded in the upward sum, with unique mappings allowing a full reconstruction of the functorial action of $\mathcal{D}$.

3.5 The Downward Category Sum

Definition 3.3. Given an equipment C and an equipment functor $\mathcal{D} : C \to \mathsf{Cat}_{\in \mathsf{Eqmnt}}$, we define their downwoard sum as the category $\exists C \mathcal{D}$ obtained by taking $\int_C \mathcal{D}$, identifying all morphism-connected objects and square-connected pro-arrows, and then reframing pro-arrows as arrows.

Pairs (c, d) are again natural in c. The natural modality takes c from the equipment C to the category $\exists C \mathcal{D}$, turning morphisms into homogeneous mappings (equalities) and pro-arrows into morphisms.

3.6 Conclusion

Self-classification is in general a bad idea. Until recently, its being common practice was justified because unimode type theory was the only type theory we had. However, we now have multimode type theory [LS16, LSR17, GKNB20], so the focus on self-classification should henceforth be met with extreme skepticism.

Part II
Jets: Higher Equipments or Directed Degrees of Relatedness

Here I will give directions that may lead to the development of a higher directed type theory with a naturality modality. Readers may want to read conclusion section 10.1.1 of my PhD thesis [Nuy20a] as introduction.

4 Higher Equipments

In this section, I will roughly develop a notion of higher equipments. The aim is not to found a new field of category theory, but to build a presheaf model for natural type theory. Therefore, I will be extremely vague about the details, especially about the operations that such a category and in particular a (higher) equipment is endowed with, because I believe they should emerge by applying the standard presheaf techniques.

4.1 Higher Equipments are n-Fold categories

Just like Cat is a 2-category, nCat is an $(n + 1)$-category. And just like we could lift Tamsamani and Simpson's condition for a double category to be a 2-category by extending Cat with profunctors, we should be able to turn nCat into an $(n + 1)$-fold category.

Note that a profunctor $\mathcal{P} : \mathcal{A} \nrightarrow \mathcal{B}$ is a *notion of heterogeneous morphisms* from objects of $\mathcal{A}$ to objects of $\mathcal{B}$. More abstractly, we will think of a pro-arrow as a thing holding a notion of arrows.

So why is Cat an equipment? Because Cat is a category (we have transformations between categories) and the pro-arrow dimension allows us to encode the categorical concept of morphisms.

Then what is Eqmnt? It is certainly a category, so we need a dimension of arrows. On top of that, we need pro-arrows encoding arrows, pro^2-arrows encoding pro-arrows, pro-squares between pro-arrows and pro^2-arrows encoding squares. Every pro-thing can be multiplied with an arrow to obtain a 'mapping of things', e.g. a square in an equipment is a mapping of arrows (living in pro-arrows). In short, we have a triple category which will probably have extra structure (at least that of an ordinary equipment). Let us call this a 3-equipment.

Then what is the category of 3-equipments 3Eqmnt? It is a quadruple category with extra structure, etc.

4.2 Mappings

Just as a profunctor holds a notion of morphisms, a functor holds a notion of mappings: $F : C \rightarrow \mathcal{D}$ allows us to state $c \mapsto_F d$. Directifying models of parametricity, we ask that everything be equipped be a mapping relation, satisfying the identity extension lemma: homogeneous mappings (mappings of the identity function) are constant, because if $x \mapsto_{\mathrm{id}} y$, then we expect that $y = \mathrm{id}(x) = x$. A consequence is that collections of anything come equipped with notions of mappings, one could call them pro-mappings, but we shall call them arrows. Thus, mappings are really pro^{-1}-arrows.

Why is Set a category? Its morphisms are notions of heterogeneous mappings of elements of sets.

4.3 Jets

Because pro^n-arrow is a really annoying word, we will instead speak of $(n + 1)$-jets. Thus, a 0-jet is a mapping, a 1-jet is a morphism, a 2-jet is a pro-arrow (side of a 2-cell), a 3-jet is a pro^2-arrow (side of a 3-cell) etc. Companions and conjoints require that, from an i-jet, we can extract an $(i + 1)$-jet in either direction.

4.4 Paths

I suggest to add, between the $(i-1)$-jet and i-jet relations, a symmetric i-path relation, and also a 0-path relation in front. There are several good reasons for doing so:

- Without these, important modalities such as parametricity/naturality will be part of *very* short chains of adjoints, and the more adjoints, the easier it is to use and to model modal type theory,
- Mixed-variance functions will have modalities that destroy i-jets, leaving only $(i+1)$-isojets behind. This will even hold for i-isojets when isojets are, in the presheaf model, simply encoded as two jets with squares witnessing mutual inversion. An i-path, however, will be respected HoTT-style.
- As a consequence of the above, parametric functions of mixed-variance types will not send 1-jets (arrows) to 0-jets (mappings) as the type does not preserve 1-jets. If we have paths, then mixed-variance types *will* send 2-paths (bridges in the sense of [NVD17], 1-edges in the sense of [ND18]) to 1-paths (isomorphisms).

  A more high-level explanation is the following: Naturality does not work too well in programming practice, so Reynolds invented relational parametricity. We are now attempting to add naturality, but this only helps us out in these rare cases that Reynolds rightfully was not satisfied with. Thus, it is not wise to throw away the parametricity infrastructure in favour of naturality.
- Others have good experience in combining directed and undirected paths in a single presheaf model [RS17, WL20]. These models, admittedly, do not have interaction between the directed and undirected paths *at the level of the base category.*

5 The Base Category

5.1 Jet Sets

We now define the category of jet sets, which in itself is not very useful, but candidate base categories for natural type theory will typically be (non-full, perhaps even non-faithful) subcategories of JetSet_d.

Definition 5.1. A depth d jet set A consists of:

- A carrier set A,
- For each $0 \leq i \leq d$, an equivalence relation $\asymp_i$ and a pre-order $\curvearrowright_i$ on A,
- Such that if $x \asymp_i y$ then $x \curvearrowright_i y$,
- Such that if $x \curvearrowright_i y$ then $x \asymp_{i+1} y$.

The category JetSet_d is defined the obvious way.

We define the following interesting jet sets of a fixed depth d:

- $\mathbb{P}_i$ is the two-element jet set generated by $0 \asymp_i 1$,
- $\mathbb{J}_i$ is the two-element jet set generated by $0 \curvearrowright_i 1$.

5.2 Features to be Considered

We will not pick one specific base category, because the experience from (cubical) HoTT teaches that there is no single 'right' one. Instead, we suggest a number of features that may help understanding the tradeoff between different choices. It is assumed that in any case, the base category of mode d will be a subcategory of JetSet_d.

- We need the path intervals $\mathbb{P}_i$ and jet intervals $\mathbb{J}_i$ to be in the base category.

- We need multipliers [ND20] for $\mathbb{P}_i$, preferrably quantifiable so that the fresh weakening be an internal modality. If we want fully fledged parametricity with the Φ/extent-rule, we probably need these multipliers to be cancellative, affine and connection-free.
- We need multipliers for $\mathbb{J}_i$. These should interact with the $\curvearrowright_i$ relation in a manner similar to Pinyo and Kraus's twisted prism functor [PK19] [Nuy20a, ex. 7.4.11]. This is necessary to obtain a contra-/covariant Hom-type that works via abstraction [Nuy20a, §10.1.1]. These multipliers are, preferrably, also quantifiable.
- It is possibly useful to be able to flip the endpoints of $\mathbb{P}_i$.
- We may want and even need to be able to extract companions and conjoints by using two inclusions of $\mathbb{J}_i$ into $\mathbb{P}_i$ (keeping and swapping the endpoints) and one or two of $\mathbb{P}_i$ into $\mathbb{J}_{i-1}$ (keeping and perhaps swapping the endpoints).
- We may want connection-like operations [CCHM17] for some or all intervals. These might in particular be used to extract the data needed to prove that the extracted companion and conjoint are indeed that: a companion and a conjoint. This seems at odds with having the Φ/extent-rule.
- We may or may not want to include non-twisted (affine, cartesian or other) products of jet intervals.
- We may or may not want to be able to extract diagonals from some or all cubes.
- The modalities in the next section should minimally be broken by the base category features. This assumes a degree of uniformity in the treatment of different path-relations, different jet-relations, and of path-relations in comparison to jet-relations.
- The base category should absolutely not be closed under colimits, because the presheaf category is the *free* cocompletion of the base category, adding new colimits even if colimits were already present. What this means is that, if $\mathcal{W}$ contains a colimit $\mathrm{colim}_i\, W_i$, then the presheaf category $\mathsf{Psh}(\mathcal{W})$ will contain its Yoneda-embedding $\mathbf{y}(\mathrm{colim}_i\, W_i)$ as well as the colimit of the Yoneda-embeddings $\mathrm{colim}_i\, \mathbf{y}(W_i)$ and for non-trivial colimits these will *never* be equal. In particular, using all of JetSet_d as the base category is a bad idea.

6 Modalities

In this section, we will simply take JetSet_d as the base category for mode d, because well-behaved choices of base categories will inherit the important modalities anyway.

6.1 Parametricity/Naturality

- The parametric/natural modality removes the 0-path (mapping of an isomorphism) and 0-jet (mapping) relations and moves forward everything that comes after. Thus, parametric/natural functions strengthen $(i + 1)$-jets/paths to i-jets/paths.
- It has a non-internalizable *left* adjoint that inserts discrete 0-path and 0-jet relations and moves back everything after.
- It has a *right* adjoint, the structural modality, that defines the new 0-path, 0-jet and 1-path (isomorphism) relations as the old 0-path relation, moving back everything else. Thus, structural functions respect 0-paths and otherwise weaken i-jets/paths to $(i + 1)$-jets/paths.
- This has a further right adjoint, that deletes the 0-jet and 1-path relations, moving forward everything beyond
- A further right adjoint defines new 0-jet and 1-path relations as the old 0-isojet relation, moving back everything beyond.

6.2 Variance

- i-Contravariance flips the i-jet relation, leaving everything else alone, and is self-inverse.
- i-Mixed-variance redefines the i-jet relation as the i-path relation. Thus, i-mixed-variant functions respect i-paths and $(i + 1)$-paths, but not i-jets. This is akin to invariance in [Nuy15].
- i-Isovariance is right adjoint to i-subvariance and redefines the i-jet relation as the $(i + 1)$-path relation. Thus, i-isovariant functions promote $(i + 1)$-paths and hence also i-jets to i-isojets. This is akin to isovariance in [Nuy15].

One might also extend the mode theory, giving the option to omit the i-jet relation if one wants to be groupoidal at that level. Note that for higher equipments, groupoidality at one level does not seem to imply groupoidality at higher levels.

6.3 Other

We might consider ad hoc polymorphism (respecting 1-paths or only 0-paths), irrelevance, shape-irrelevance, ...

7 Fibrancy

The following notions of fibrancy (with suggested definitions) seem important:

0-discreteness Ideally, all types have a discrete 0-path and 0-jet relation: homogeneous 0-paths and 0-jets should be constant. 0-discreteness would then have to be proven in the model.

Functoriality This is usually called covariance, but in the envisioned type system, variance is handled by modalities while the actual functorial action should be asserted by fibrancy. Basically, I expect that one should formulate a (composition and) transport operator as exists in cubical HoTT, where transport is always along a 0-jet dimension in the proper direction. (Transport along 0-paths need not be added explicitly *provided that* companions and conjoints and their witnesses can be extracted from paths.)

Kan Kan types are types where we can compose 1-paths (isomorphisms). A type is Kan if its path type (defined by applying the naturality modality to the $\forall(i : \mathbb{P}_k)$-type [Nuy20a, §10.1.1]) is functorial.

Segal Segal types are types where we can compose 1-jets (arrows). A type is Segal if its Hom-type [Nuy20a, §10.1.1] is functorial. For reasons explained briefly in [Nuy18] and at length in [Nuy20a, ch. 8], we need to consider Segal fibrancy, and hence also functoriality, *contextually*. Contextual fibrancy can hopefully be considered internally without special judgemental support by making use of embargo multipliers [Nuy20b].

Rezk Rezk types are types where the extraction of a 1-isojet (pair of mutually inverse 1-jets) from a 1-path is invertible.

In absence of conjunction-like operations, we may want to impose the equipment conditions.

8 Universes

The universe of functorial types is obtained by applying the left adjoint to parametricity to the Hofmann-Streicher universe [HS97] of functorial types. Decoding types is then a parametric/natural operation (see also [Nuy20a, ch. 9]), leading effectively to a distinction between the modality of a term and its type's code as in [NVD17, ND18].

9 Conclusion

I have sketched how to build a higher directed type system with a modality for naturality. The work ahead is formidable and it seems to me that the first step is to devise a better type system for working with multipliers and the transpension type. A first exploration of natural type theory as described above is probably best done using a base category where companions, conjoints and connections can be extracted, accepting that parametricity will have to be done via Glue, Weld and Ψ/Gel rather than via the Φ/extent-rule [ND20].

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