Parametric Quantifiers for Dependent Type Theory

Andreas Nuyts*, Andrea Vezzosi‡, Dominique Devriese*

*KU Leuven, ‡Chalmers University of Technology

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Overview

Parametricity

- Type variable is **parametric** if **only used for type-checking** $\Rightarrow$ free well-behavedness theorems.
- Well-studied in System F, System Fω, Haskell, ...

Parametricity in dependent types

- **Some** parametricity results carry over,
  Takeuti (2001), Bernardy, Jansson and Paterson (2012), Krishnaswami and Dreyer (2013), Atkey, Ghani and Johann (2014)
- Some can be made provable **internally**,
  Bernardy, Coquand and Moulin (2015), Guilhem Moulin's PhD (2016)
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- We carry over "the" **remaining theorems** metatheoretically.
- We allow proving **additional theorems internally**.

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Parametricity, intuitively

In System F, Fω, Haskell, ..., **type parameters** are parametric.

- Only used for type-checking,
- Not inspected (e.g. no pattern matching),
- Same algorithm on all types.

Enforced by the type system.

Example

$$\text{flatten} \quad : \quad \forall X.\text{Tree } X \to \text{List } X$$

By parametricity:

$$\begin{tikzcd} A \arrow[d, "f"'] & \text{Tree } A \arrow[r, "\text{flatten}"] \arrow[d, "\text{Tree } f"'] & \text{List } A \arrow[d, "\text{List } f"] \\ B & \text{Tree } B \arrow[r, "\text{flatten}"] & \text{List } B \end{tikzcd}$$

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Parametricity: Representation independence

Theorem

$$(A \to B) \cong \left(\underbrace{\forall X.(X \to A)}_{\text{For any representation } (X, r) \text{ of } A} \to (X \to B) \right)$$

Proof:

($\to$) $h \mapsto \lambda X.\lambda r.h \circ r.$

($\leftarrow$) $g \mapsto g\ A\ \mathrm{id}.$

(src) refl

(tgt) Prove: $g\ X\ r\ x = g\ A\ \mathrm{id}\ (r\ x).$

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Relational parametricity

Lemma

If $g : \forall X.(X \to A) \to (X \to B)$
then $g\ X_0\ r_0\ x_0 = g\ A\ \mathrm{id}\ (r_0\ x_0)$.

Rel. param.: A sound scheme for proving parametricity theorems.
Idea: **Related things map to related things.**

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Relational parametricity

This is a **metatheoretical** scheme for System F, System Fω, ...

- Can we do this for **dependent types**?
- Can we do this **internally** in dependent types?

Π is not parametric

System F:
$\forall X.(X \to A) \to (X \to B)$.

Dependent types:
$\Pi(X : \mathcal{U}).(X \to A) \to (X \to B)$.

Suppose $B = \mathcal{U}$:
$leak : \Pi(X : \mathcal{U}).(X \to A) \to (X \to \mathcal{U})$
$leak\ X\ r\ x = X$.
Representation type is returned as data!

In existing work: $\mathcal{U}$ violates identity extension lemma (IEL).
Takeuti (2001), Bernardy, Jansson and Paterson (2012), Krishnaswami and Dreyer (2013), Atkey, Ghani and Johann (2014)

Our solution (syntax-side): Keep track of how we use variables.

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Adding parametric quantifiers

Non-parametric quantifiers

Parametric quantifiers

Non-parametric functions

$$f : \Pi(x : A).B\,x, \qquad f : A \to B$$

can use argument **as data**.

$$\frac{A : \mathcal{U} \qquad B : A \to \mathcal{U}}{\Pi(x : A).B\,x : \mathcal{U}}$$

Parametric functions

$$f : \forall(x : A).B\,x$$

cannot inspect argument: it is used **only for type-checking**.*

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Non-parametric pairs

$$p : \Sigma(x : A).B\,x, \qquad p : A \times B$$

Parametric pairs (packs)

$$p : \exists(x : A).B\,x$$

*if all elements of A are related.

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Use $\forall$ if you want representation independence:
$\forall(X : \mathcal{U}).(X \to A) \to (X \to B)$.

The following is now ill-typed:
$\mathit{leak} : \forall(X : \mathcal{U}).(X \to A) \to (X \to \mathcal{U})$
$\mathit{leak}\ X\ r\ x = X$,
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- Graph type of $r : X \to A$
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The framework:

- Type system **ParamDTT** with Π and Σ, $\forall$ and $\exists$,
- **Soundness** using 'bridge/path cubical sets' (higher-dimensional labelled reflexive graphs),
- We **extend Agda** with support for ParamDTT,

Results:

- **Stronger internal** parametricity system,
  - but not (yet?) fully iterated,
- We show **internally** that **Church encoding** of data (e.g. lists) and codata (e.g. streams) works,
  - up to predicativity issues,
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Thanks!

Related talks:

Normalization by Evaluation for Sized Dependent Types
Abel, Vezzosi, Winterhalter – Up next

A Fibrational Framework for Substructural and Modal Logics
Licata, Shulman, Riley – 13h @ FSCD

Questions?

Church initial algebras

Assume level-preserving functor F.

$\mathsf{Mu}_\ell = \forall(X : \mathcal{U}_\ell).(FX \to X) \to X.$
$\mathsf{mkMu}_\ell : F\mathsf{Mu}_\ell \to \mathsf{Mu}_\ell.$

$\mathsf{fold}_\ell\ A\ \mathsf{mk}A = \lambda m.(m\ A\ \mathsf{mk}A) : \mathsf{Mu}_\ell \to A$

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Theorem (Initiality of Mu up to $\downarrow$)

For any B, $\mathsf{mk}B$ and any algebra morphism $f : \mathsf{Mu}_\ell \to B$:

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Initial algebras using Size

To build a fixpoint List A of $(\mathrm{Unit} + A \times \sqcup)$:

- By well-founded induction on n : Size, build $\widehat{\mathrm{List}}\ A\ n \cong \mathrm{Unit} + A \times (\exists m < n.\widehat{\mathrm{List}}\ A\ m)$,
  - Special fixpoint operator for Size.
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Parametricity: side bounds are hidden

Works for finitely branching container functors (even indexed): $F\ T = \Sigma(c : C).(B\ c \to T)$.
Also final co-algebras (e.g. streams).

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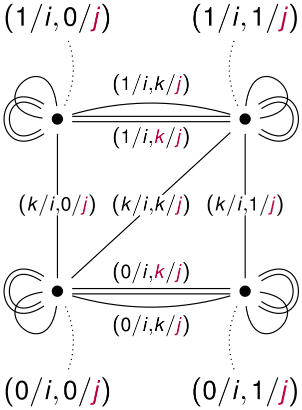
Example of a bridge/path cubical set

The context $(i :^{\sqcap} \mathbb{I}, j :^{\forall} \mathbb{I})$ as a bridge/path cubical set.

legend:



Related work

citation	source	target	journey	model	IEL proof
Reynolds, 1983	Sys F			set th.	**yes**
Abadi, Cardelli, Curien, 1993	Sys F	Sys $\mathscr{R}$	external		**yes**
Plotkin & Abadi, 1993	Sys F	Sys F + logic	external		**yes**
Wadler, 2007	Sys F	Sys F + logic	external		**yes**
Takeuti, 2001	$\mathscr{X} \in \lambda$-cube	$\mathscr{Y} \in \lambda$-cube	external		for small types
Bernardy, Jansson, Paterson, 2012	any PTS	other PTS	external		no
Krishnaswami & Dreyer, 2013	dependent types			Q-PER	only some corollaries
Atkey, Ghani, Johann, 2014	dependent types			presh.	for small types
Bernardy, Coquand, Moulin, 2015	dependent types + param. operators	same as source	**internal**	presh.	no
This work	dependent types + param. operators, $\forall$, $\exists$	same as source	**internal**	presh.	**yes**

Parametricity, Shape-irrelevance, irrelevance

4 functors on bridge/path cubical sets:

- id Non-parametricity (continuity)
- $\sharp$ Parametricity
- $\bullet\bullet$ Shape-irrelevance
- $\bullet$ Irrelevance

such that $\sharp \circ (\bullet\bullet) = \bullet$.

Abel et al. have: $[\![\text{Size}]\!] = [\![\mathbb{N}]\!]$.
We have: $[\![\text{Size}]\!] = \bullet\bullet [\![\mathbb{N}]\!]$. Hence, $\sharp [\![\text{Size}]\!] = \bullet [\![\mathbb{N}]\!]$.

ParamDTT	Abel	$[\![\text{domain}]\!]$
$\Pi(i : \text{Size}).A\ i$	$\bullet\bullet(i : \text{Size}) \to A\ i$	$\bullet\bullet [\![\mathbb{N}]\!]$
$\forall(i : \text{Size}).A\ i$	$\bullet(i : \text{Size}) \to A\ i$	$\bullet [\![\mathbb{N}]\!]$