Internalizing Presheaf Semantics: Charting the Design Space

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LICS '18
Oxford, UK
July 7, 2018

Presheaf semantics can model:

- **Parametricity** (preservation of **relations**),
- **HoTT** (preservation of **equivalences**),
- **Directed TT** (preservation of **homomorphisms**).

Operators for **cheap proofs** of **free preservation theorems**?

- Cubical TT: Glue
- NVD17, ND18: Glue, Weld
- Moulin (PhD on internal param'ty): Ψ, Φ
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Universal Type Extension Operators (Glue, Weld)

Final extension of (T, f)

$$\frac{\Gamma \vdash A\ \textbf{type} \qquad \Gamma \vdash P\ \textbf{prop} \qquad \Gamma, P \vdash T\ \textbf{type} \qquad \Gamma, P \vdash f : T \to A}{\Gamma \vdash \textbf{Glue}\,\{A \leftarrow (P\,?\,T, f)\}\ \textbf{type} \qquad \Gamma, P \vdash \textbf{Glue}\,\{A \leftarrow (P\,?\,T, f)\} = T}$$

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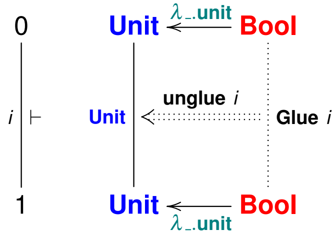

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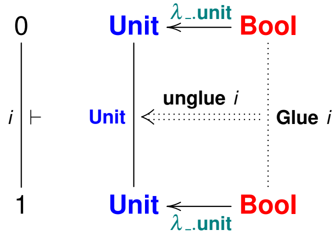
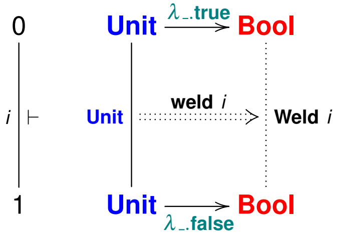


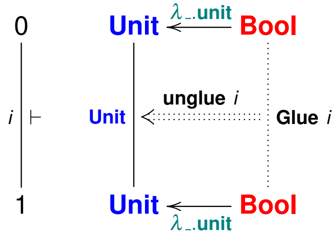


Final extension of (T, f)

$$\frac{\begin{array}{c}\Gamma \vdash \textcolor{blue}{A}\ \textbf{type} \\ \Gamma \vdash P\ \textbf{prop} \\ \Gamma, P \vdash \textcolor{red}{T}\ \textbf{type} \\ \Gamma, P \vdash f : \textcolor{red}{T} \to \textcolor{blue}{A}\end{array}}{\begin{array}{c}\Gamma \vdash \textbf{Glue}\,\{\textcolor{blue}{A} \leftarrow (P\,?\,\textcolor{red}{T}, f)\}\ \textbf{type} \\ \Gamma, P \vdash \textbf{Glue}\,\{\textcolor{blue}{A} \leftarrow (P\,?\,\textcolor{red}{T}, f)\} = \textcolor{red}{T}\end{array}}$$

Theorem (Cubical TT)

$$\frac{\begin{array}{c}\textcolor{blue}{A}\ \text{Kan} \\ \textcolor{red}{T}\ \text{Kan} \\ f\ \text{equivalence}\end{array}}{\textbf{Glue}\ \text{Kan}}$$



As (co-)inductive types

Final extension of (T, f)

record $G :=$ **Glue** $\{A \leftarrow (P\,?\,T, f)\}$
where

- **unglue** : $G \to A$
- **reduce** : $G \to (\_ : P) \to T$
- **coh** : $(g : G) \to (\_ : P) \to$
  f (**reduce** g _) $\equiv_A$ **unglue** g

G extends T
unglue extends f
reduce is id_T
coh is **refl**

Initial extension of (T, f)

HIT $W :=$ **Weld** $\{A \to (P\,?\,T, f)\}$
where

- **weld** : $A \to W$
- **include** : $(\_ : P) \to T \to W$
- **coh** : $(a : A) \to (\_ : P) \to$
  include $(f\ a) \equiv_W$ **weld** a

W extends T
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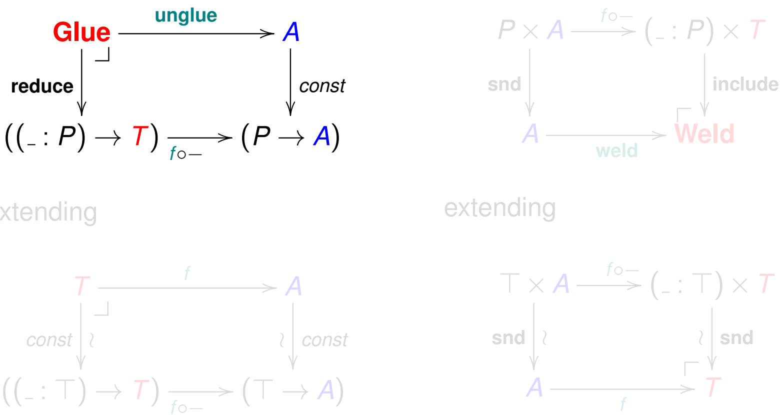
As pullback/pushout



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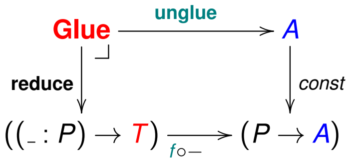

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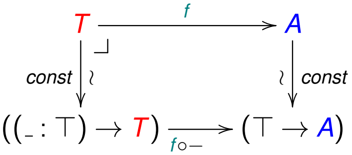

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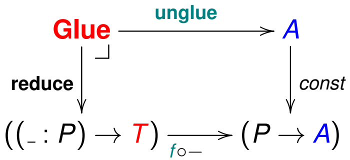

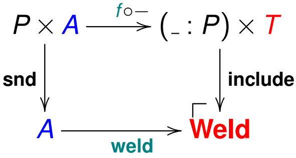

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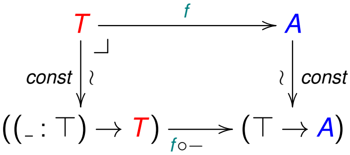

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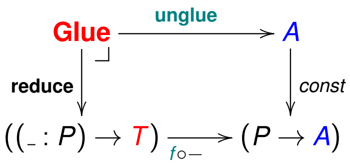

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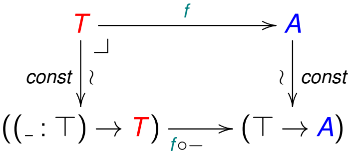

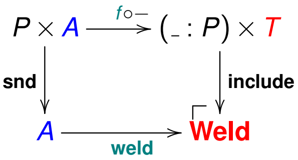

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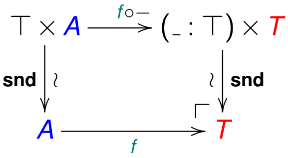

A milling cutter

NL: frees

FR: fraise

DE: Fräser

PL: frez

JP: furaisu

CMN: xǐdāo

If base category has products;
for any **shape** $\mathbb{I}$:

$$\begin{array}{ccc} \Pi\mathbb{I}\,P \times \Pi\mathbb{I}\,A & \longrightarrow & (\_ : \Pi\mathbb{I}\,P) \times \Pi\mathbb{I}\,T \\ \downarrow & & \downarrow \\ \Pi\mathbb{I}\,A & \longrightarrow & \Pi\mathbb{I}\,\mathbf{Weld} \end{array}$$

$$\mathbf{mill} :$$
$$(\Pi i.\mathbf{Weld}\,\{A\,i \to (P\,i\,?\,T\,i, f\,i)\})$$
$$\cong$$
$$\mathbf{Weld}\,\{\Pi i.A\,i \to (\forall i.P\,i\,?\,\Pi i.T\,i, f \circ -)\}$$

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Debating nomenclature

Glue glue unglue	Weld weld
Glue glue unglue	Coglue co**un**glue
FExt ? ?	IExt ?

Different name in Cubical TT?

Debating nomenclature

Glue	Weld
glue	weld
unglue	
Glue	Coglue
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?	?
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Glue/Weld: Summary

Universal type extension operators (Glue/Weld):

- Exist in any presheaf model,
- Internalize nothing about the **particular** model.

Combine with something else:

- Box filling (Cubical TT)
- Modalities & identity extension lemma [NVD17]
- **mill** (identifies **shape** types for particular model)

Boundary-Filling Operators
(Φ, Ψ)

Generalized from:
Bernardy, Coquand, Moulin (2015)
Moulin's PhD (2016)

Boundaries

Definition (Boundary)

For any shape $\mathbb{I} \in \mathcal{C}$,
the **boundary** is the greatest strict subobject $\partial\mathbb{I} \subsetneq \mathbf{y}\mathbb{I} \in \widehat{\mathcal{C}}$.

Theorem

$$(\mathbf{y}\mathbb{U} \to \partial\mathbb{I}) \cong (\mathbb{U} \to \mathbb{I}) \setminus \{\text{split epis}\}.$$

Note:

$\varphi : \mathbb{U} \to \mathbb{V}$ split epi $\quad\Leftrightarrow\quad$ $\mathbf{y}\varphi : \mathbf{y}\mathbb{U} \to \mathbf{y}\mathbb{V}$ epi
$\varphi : \mathbb{U} \to \mathbb{V}$ mono $\quad\Leftrightarrow\quad$ $\mathbf{y}\varphi : \mathbf{y}\mathbb{U} \to \mathbf{y}\mathbb{V}$ mono

Example

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Fillers

$$\frac{\Gamma, i : \mathbb{I} \vdash A\,\textbf{type} \qquad \Gamma, i : \partial\mathbb{I} \vdash a : A}{\Gamma \vdash \textbf{Filler}_{i.A}\,(i.a)\,\textbf{type}}$$

$$\textbf{Filler}_{i.A}\,(i.a) \cong (f : (i : \mathbb{I}) \to A) \times ((i : \partial\mathbb{I}) \to f\ i =_A a)$$

Example (Cubical TT)

$\textbf{Filler}_{i.A}\,(i.a) = \textbf{Path}_{i.A}(a[0/i], a[1/i])$.

Boundary-filling

For any shape $\mathbb{I}$:

$$\frac{\Gamma \vdash f_\partial : (i : \partial\mathbb{I}) \multimap (x : A\ i) \to B\ i\ x \qquad \Gamma \vdash h : (\xi : (i : \mathbb{I}) \multimap A\ i) \to \mathbf{Filler}_{i.B\ i\ (\xi\ i)}\ (f_\partial\ i\ (\xi\ i))}{\Gamma \vdash \Phi(f_\partial, h) : (i : \mathbb{I}) \multimap (x : A\ i) \to B\ i\ x}$$

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Compare: funext and univalence

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Problem with diagonals

How to build
$f : (i, j : \mathbb{I}) \multimap (x : A\ i\ j) \to B\ i\ j\ x$?
$f =$
$\Phi^2(f_{00}, f_{01}, f_{10}, f_{11}, h_0, h_1, k_0, k_1, w)$

f_{00} f_{01}

w

f_{10} f_{11}

- $f\ 0\ 0\ a = f_{00}\ a$
- $f\ 0\ j\ a = k_0\ (\lambda j.a)\ j$
- $f\ i\ j\ a = w\ (\lambda i.\lambda j.a)\ i\ j$
- $f\ i\ i\ a = ?\ (\lambda i.a)\ i$

Solution:
Base category: $\mathbb{I} \not\to \mathbb{I} * \mathbb{I}$
Separated product: (cf. nom. sets)
$[\![\Gamma, i : \mathbb{I}]\!] = [\![\Gamma]\!] * \mathbf{y}\mathbb{I}$

"Linear" application:

$$\frac{\Gamma \vdash f : (i : \mathbb{I}) \multimap A\ i}{\Gamma, i : \mathbb{I} \vdash f\ i : A\ i}$$

Incorporating Φ (J-P. Bernardy):

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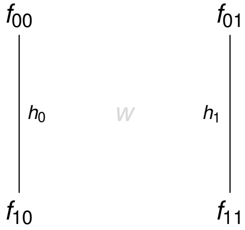

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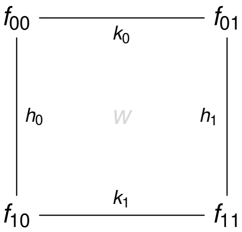

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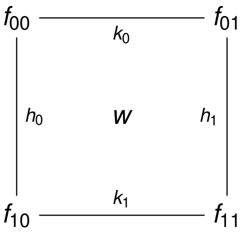

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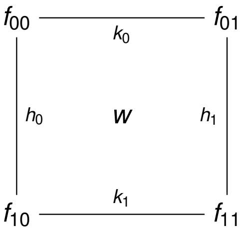

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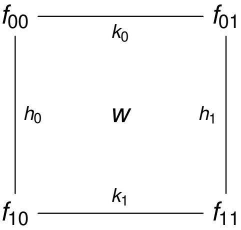

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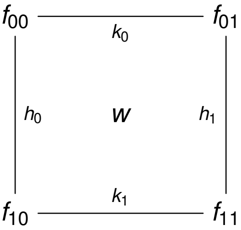

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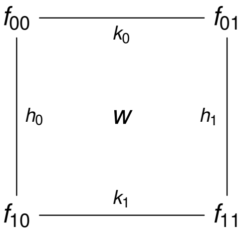

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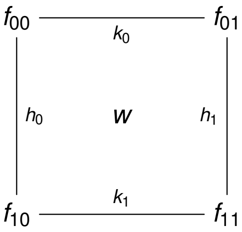

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Freshness predicate

Exchange only works one way.

$$(\Gamma, a : A, i : \mathbb{I}, \Delta)$$
$$\cong$$
$$(\Gamma, i : \mathbb{I}, a : A, i \# a, \Delta)$$
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"a does not vary with i."

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"*a* does not vary with *i*."

Details on semantics

Definition (Suitable base category)

- sym. **monoidal** with **terminal unit**,
- projection $(\mathbb{U} * \mathbb{V}) \to \mathbb{V}$ cartesian on monos,
- generalized Reedy w.r.t. **split epis** and **monos** (can be relaxed)
- all $\mathbb{U} \to \top$ split epi, equiv.: $\partial\top = \varnothing$

Definition

$\mathcal{C}//\mathbb{U}$: split epi slices.

Theorem

$(\mathbb{U} * -, \pi_1) : \mathcal{C} \to \mathcal{C}//\mathbb{U}$ is **faithful**.

Definition (Diagonal-free)

$(\mathbb{U} * -, \pi_1) : \mathcal{C} \to \mathcal{C}//\mathbb{U}$ is **full**.
Rules out
$\delta : (\mathbb{U} * \top, \pi_1) \not\to (\mathbb{U} * \mathbb{U}, \pi_1)$

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Definition (Cartesian)

$* = \times$

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	Glue	**Weld**	**mill**	Φ	Ψ	$\#$
mon. base cat.			•	(•)	(•)	(•)
suit. base cat.				•	•	•
cartesian						
diag.-free				•?	•?	•
conn.-free				•	•[1]	•

[1] With connections: Ψ is sound but underspecified.

Results

Theorem

$$\Psi, \Phi, \text{colimit systems} \models \textbf{Glue}, \textbf{Weld}, \textbf{mill}$$

(where P ranges only over a shape $\mathbb{U}$)

Sketch of proof: By induction on Reedy-degree of $\mathbb{U}$

- Define **Glue**/**Weld**/**mill** on $\partial\mathbb{U}$
  $\partial\mathbb{U} = \textbf{colim}_i \mathbb{V}_i$ ($\deg \mathbb{V}_i < \deg \mathbb{U}$)
  IH: defined on $\mathbb{V}_i$
  Colimit system: paste together for $\partial\mathbb{U}$
- Fill the boundary using Φ/Ψ. □

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$$\mathbf{Weld}, \mathbf{mill}, \# \models \Psi$$

Sketch of proof:

$\Psi(A_\partial, P) : \mathbb{I} \multimap \mathcal{U}$

$\Psi(A_\partial, P)\ i = \mathbf{Weld}\,\{A_\Sigma\ i \to (i \in \partial\mathbb{I}\ ?\ A_\partial\ i, f\ i)\}$

$((i : \mathbb{I}) \multimap \Psi(A_\partial, P)\ i)$

$=_{\mathrm{def}} ((i : \mathbb{I}) \multimap \mathbf{Weld}\,\{A_\Sigma\ i \to (i \in \partial\mathbb{I}\ ?\ A_\partial\ i, f\ i)\})$

$\cong_{\mathbf{mill}} \mathbf{Weld}\,\{((i : \mathbb{I}) \multimap A_\Sigma\ i) \to (\bot\ ?\ \lightning, \lightning)\}$

$\cong_{\mathbf{ind_{Weld}}} ((i : \mathbb{I}) \multimap A_\Sigma\ i)$

$\cong_{\mathrm{wanted}} (a_\partial : (j : \partial\mathbb{I}) \multimap A_\partial\ j) \times (p : P\ a_\partial)$

$A_\Sigma : \mathbb{I} \multimap \mathcal{U}$

$A_\Sigma\ i = (a_\partial : (j : \partial\mathbb{I}) \multimap A_\partial\ j) \times (p : P\ a_\partial) \times ((a_\partial, p) \# i)$

$\Rightarrow \quad ((i : \mathbb{I}) \multimap A_\Sigma\ i) \cong (a_\partial : (j : \partial\mathbb{I}) \multimap A_\partial\ j) \times (p : P\ a_\partial)$

$f : (i : \partial\mathbb{I}) \multimap A_\Sigma\ i \to A_\partial\ i$

$f\ i\ (a, p, \_) = a\ i$ $\square$

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$$\textbf{Weld}, \textbf{mill}, \# \models \Psi$$

Sketch of proof:

$\Psi(A_\partial, P) : \mathbb{I} \multimap \mathcal{U}$

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Theorem

$$\textbf{Glue}, \textbf{Weld}, \textbf{mill}, \Psi, \# \not\models \Phi$$

Sketch of proof: Pick fully faithful functor $I : \mathcal{D} \to \mathcal{C}$.
$\widehat{\mathcal{C}} \models \textbf{Glue}_{\mathcal{D}}, \textbf{Weld}_{\mathcal{D}}, \textbf{mill}_{\mathcal{D}}, \Psi_{\mathcal{D}}, \#_{\mathcal{D}}$,
$\widehat{\mathcal{C}} \not\models \Phi_{\mathcal{D}}$ (in general, e.g. ∇ : **Cube** $\to$ **BPCube**)
because $\Phi_{\mathcal{D}}(f_\partial, h)$ has no action on $\mathbb{U}$-cells for $\mathbb{U} \in \mathcal{C} \setminus I(\mathcal{D})$. □

$$[\![-]\!] : \{\text{System F types}\} \to \{\text{MLTT types}\}$$

Theorem

$\Phi_{\textbf{Cube}}, \Psi_{\textbf{Cube}} \models$ Every term $t : [\![T]\!]$ is parametric.

Sketch of proof:
use Ψ to convert $(A_0, A_1, \bar{A})$ to $A : \mathbb{I} \multimap \mathcal{U}$,
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Proof: BPCube models LHS, not RHS. □

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Conclusion

Cartesian Φ and Ψ would be best. (Working on it.)
Alas: they don't play well with connections.

Thanks!

Questions?