Robust Notions of Contextual Fibrancy

Andreas Nuyts

KU Leuven, Belgium

Workshop on HoTT/UF '18
Oxford, UK
July 8, 2018

Problem statement

Cat. of contexts	Cubical sets	Simplicial sets	Cubical sets
Notion of fibrancy	Kan	Segal	Discreteness
Gen. left maps	$\sqcup \subseteq \square$	spine $\subseteq$ simplex ($\exists!$)	$\Phi \times \mathbb{I} \to \Phi$
Closed fibrant types?	∞-Groupoids	Categories	Sets
Π_A preserves fibrancy?	**if *A* fibrant**	**if *A* Conduché** "composite $\subseteq$ simplex"	**YES**
Fib. repl. commutes with substitution?	**NO**	**NO**	**YES**

How to get **YES**ses everywhere?

Problem statement

Cat. of contexts	Cubical sets	Simplicial sets	Cubical sets
Notion of fibrancy	Kan	Segal	Discreteness
Gen. left maps	$\sqcup \subseteq \square$	spine $\subseteq$ simplex ($\exists!$)	$\Phi \times \mathbb{I} \to \Phi$
Closed fibrant types?	∞-Groupoids	Categories	Sets
Π_A preserves fibrancy?	**if A fibrant**	**if A Conduché** "composite $\subseteq$ simplex"	**YES**
Fib. repl. commutes with substitution?	**NO**	**NO**	**YES**

How to get **YES**ses everywhere?

Problem statement

Cat. of contexts	Cubical sets	Simplicial sets	Cubical sets
Notion of fibrancy	Kan	Segal	Discreteness
Gen. left maps	$\sqcup \subseteq \square$	spine $\subseteq$ simplex ($\exists!$)	$\Phi \times \mathbb{I} \to \Phi$
Closed fibrant types?	∞-Groupoids	Categories	Sets
Π_A preserves fibrancy?	**if A fibrant**	**if A Conduché** "composite $\subseteq$ simplex"	**YES**
Fib. repl. commutes with substitution?	**NO**	**NO**	**YES**

How to get **YES**ses everywhere?

Problem statement

Cat. of contexts	Cubical sets	Simplicial sets	Cubical sets
Notion of fibrancy	Kan	Segal	Discreteness
Gen. left maps	$\sqcup \subseteq \square$	spine $\subseteq$ simplex ($\exists!$)	$\Phi \times \mathbb{I} \to \Phi$
Closed fibrant types?	∞-Groupoids	Categories	Sets
Π_A preserves fibrancy?	**if *A* fibrant**	**if *A* Conduché** "composite $\subseteq$ simplex"	**YES**
Fib. repl. commutes with substitution?	**NO**	**NO**	**YES**

How to get **YES**ses everywhere?

Problem statement

Cat. of contexts	Cubical sets	Simplicial sets	Cubical sets
Notion of fibrancy	Kan	Segal	Discreteness
Gen. left maps	$\sqcup \subseteq \square$	spine $\subseteq$ simplex ($\exists!$)	$\Phi \times \mathbb{I} \to \Phi$
Closed fibrant types?	∞-Groupoids	Categories	Sets
Π_A preserves fibrancy?	**if *A* fibrant**	**if *A* Conduché** “composite $\subseteq$ simplex”	**YES**
Fib. repl. commutes with substitution?	**NO**	**NO**	**YES**

How to get **YES**ses everywhere?

Problem statement

Cat. of contexts	Cubical sets	Simplicial sets	Cubical sets
Notion of fibrancy	Kan	Segal	Discreteness
Gen. left maps	$\sqcup \subseteq \square$	spine $\subseteq$ simplex ($\exists!$)	$\Phi \times \mathbb{I} \to \Phi$
Closed fibrant types?	∞-Groupoids	Categories	Sets
Π_A preserves fibrancy?	**if A fibrant**	**if A Conduché** "composite $\subseteq$ simplex"	**YES**
Fib. repl. commutes with substitution?	**NO**	**NO**	**YES**

How to get **YES**ses everywhere?

Problem statement

Cat. of contexts	Cubical sets	Simplicial sets	Cubical sets
Notion of fibrancy	Kan	Segal	Discreteness
Gen. left maps	$\sqcup \subseteq \square$	spine $\subseteq$ simplex ($\exists!$)	$\Phi \times \mathbb{I} \to \Phi$
Closed fibrant types?	∞-Groupoids	Categories	Sets
Π_A preserves fibrancy?	**if A fibrant**	**if A Conduché** “composite $\subseteq$ simplex”	**YES**
Fib. repl. commutes with substitution?	**NO**	**NO**	**YES**

How to get **YES**ses everywhere?

Problem statement

Cat. of contexts	Cubical sets	Simplicial sets	Cubical sets
Notion of fibrancy	Kan	Segal	Discreteness
Gen. left maps	$\sqcup \subseteq \square$	spine $\subseteq$ simplex ($\exists!$)	$\Phi \times \mathbb{I} \to \Phi$
Closed fibrant types?	∞-Groupoids	Categories	Sets
Π_A preserves fibrancy?	**if A fibrant**	**if A Conduché** "composite $\subseteq$ simplex"	**YES**
Fib. repl. commutes with substitution?	**NO**	**NO**	**YES**

How to get **YES**ses everywhere?

Problem statement

Cat. of contexts	Cubical sets	Simplicial sets	Cubical sets
Notion of fibrancy	Kan	Segal	Discreteness
Gen. left maps	$\sqcup \subseteq \square$	spine $\subseteq$ simplex ($\exists!$)	$\Phi \times \mathbb{I} \to \Phi$
Closed fibrant types?	∞-Groupoids	Categories	Sets
Π_A preserves fibrancy?	**if *A* fibrant**	**if *A* Conduché** “composite $\subseteq$ simplex”	**YES**
Fib. repl. commutes with substitution?	**NO**	**NO**	**YES**

How to get **YES**ses everywhere?

Problem statement

Cat. of contexts	Cubical sets	Simplicial sets	Cubical sets
Notion of fibrancy	Kan	Segal	Discreteness
Gen. left maps	$\sqcup \subseteq \square$	spine $\subseteq$ simplex ($\exists!$)	$\Phi \times \mathbb{I} \to \Phi$
Closed fibrant types?	∞-Groupoids	Categories	Sets
Π_A preserves fibrancy?	**if *A* fibrant**	**if *A* Conduché** "composite $\subseteq$ simplex"	**YES**
Fib. repl. commutes with substitution?	**NO**	**NO**	**YES**

How to get **YES**ses everywhere?

Problem statement

Cat. of contexts	Cubical sets	Simplicial sets	Cubical sets
Notion of fibrancy	Kan	Segal	Discreteness
Gen. left maps	$\sqcup \subseteq \square$	spine $\subseteq$ simplex ($\exists!$)	$\Phi \times \mathbb{I} \to \Phi$
Closed fibrant types?	∞-Groupoids	Categories	Sets
Π_A preserves fibrancy?	**if *A* fibrant**	**if *A* Conduché** "composite $\subseteq$ simplex"	**YES**
Fib. repl. commutes with substitution?	**NO**	**NO**	**YES**

How to get **YES**ses everywhere?

Problem statement

Cat. of contexts	Cubical sets	Simplicial sets	Cubical sets
Notion of fibrancy	Kan	Segal	Discreteness
Gen. left maps	$\sqcup \subseteq \square$	spine $\subseteq$ simplex ($\exists!$)	$\Phi \times \mathbb{I} \to \Phi$
Closed fibrant types?	∞-Groupoids	Categories	Sets
Π_A preserves fibrancy?	**if *A* fibrant**	**if *A* Conduché** “composite $\subseteq$ simplex”	**YES**
Fib. repl. commutes with substitution?	**NO**	**NO**	**YES**

How to get **YES**ses everywhere?

Kan fibrancy of Π

Bezem, Coquand & Huber (2014), Huber's Lic/PhD (2015/2016)

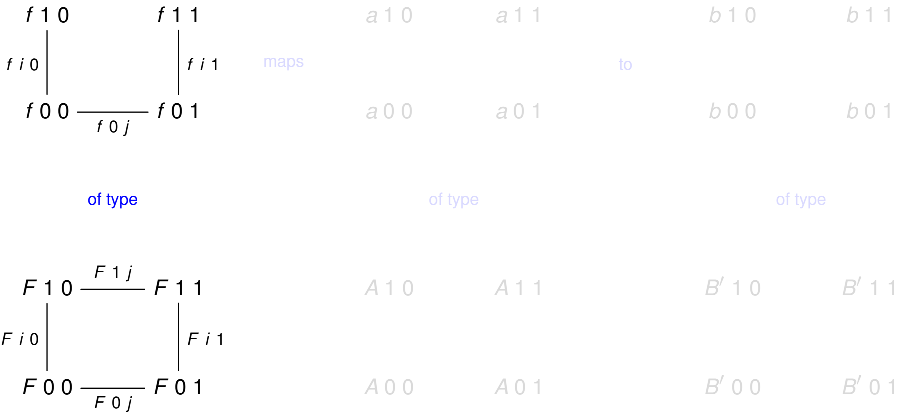
Abbreviate

- $b\ i\ j := f\ i\ j\ (a\ i\ j)$
- $F\ i\ j := (x : A\ i\ j) \to B\ i\ j\ x$
- $B'\ i\ j := B\ i\ j\ (a\ i\ j)$

Bezem, Coquand & Huber (2014), Huber's Lic/PhD (2015/2016)

Abbreviate

- $b\ i\ j := f\ i\ j\ (a\ i\ j)$
- $F\ i\ j := (x : A\ i\ j) \to B\ i\ j\ x$
- $B'\ i\ j := B\ i\ j\ (a\ i\ j)$



Kan fibrancy of Π

Bezem, Coquand & Huber (2014), Huber's Lic/PhD (2015/2016)

Abbreviate

- $b\ i\ j := f\ i\ j\ (a\ i\ j)$
- $F\ i\ j := (x : A\ i\ j) \to B\ i\ j\ x$
- $B'\ i\ j := B\ i\ j\ (a\ i\ j)$

$$\begin{array}{ccc} f\ 1\ 0 & \overset{f\ 1\ j}{-\,-\,-} & f\ 1\ 1 \\ {\scriptstyle f\ i\ 0}\Big| & & \Big|{\scriptstyle f\ i\ 1} \\ f\ 0\ 0 & \underset{f\ 0\ j}{\rule{2em}{0.4pt}} & f\ 0\ 1 \end{array} \quad \text{maps} \quad \begin{array}{cc} a\ 1\ 0 & a\ 1\ 1 \\ & \\ a\ 0\ 0 & a\ 0\ 1 \end{array} \quad \text{to} \quad \begin{array}{cc} b\ 1\ 0 & b\ 1\ 1 \\ & \\ b\ 0\ 0 & b\ 0\ 1 \end{array}$$

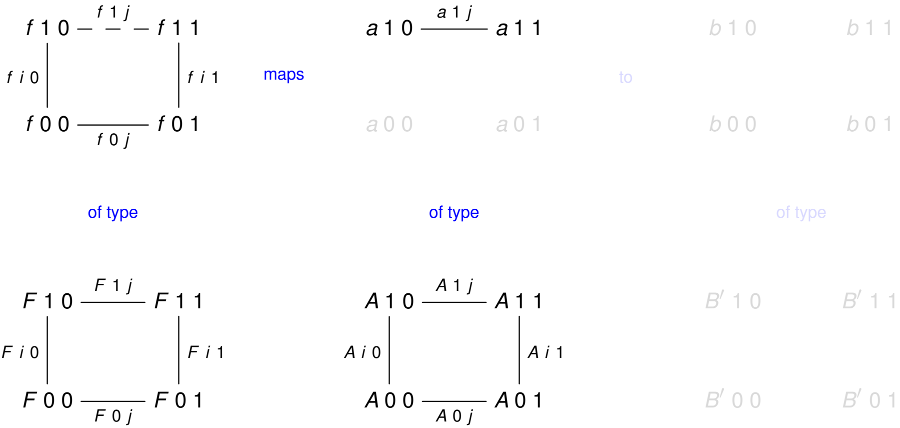
of type | of type | of type

$$\begin{array}{ccc} F\ 1\ 0 & \overset{F\ 1\ j}{\rule{2em}{0.4pt}} & F\ 1\ 1 \\ {\scriptstyle F\ i\ 0}\Big| & & \Big|{\scriptstyle F\ i\ 1} \\ F\ 0\ 0 & \underset{F\ 0\ j}{\rule{2em}{0.4pt}} & F\ 0\ 1 \end{array} \qquad \begin{array}{cc} A\ 1\ 0 & A\ 1\ 1 \\ & \\ A\ 0\ 0 & A\ 0\ 1 \end{array} \qquad \begin{array}{cc} B'\ 1\ 0 & B'\ 1\ 1 \\ & \\ B'\ 0\ 0 & B'\ 0\ 1 \end{array}$$

Bezem, Coquand & Huber (2014), Huber's Lic/PhD (2015/2016)

Abbreviate

- $b\ i\ j := f\ i\ j\ (a\ i\ j)$
- $F\ i\ j := (x : A\ i\ j) \to B\ i\ j\ x$
- $B'\ i\ j := B\ i\ j\ (a\ i\ j)$

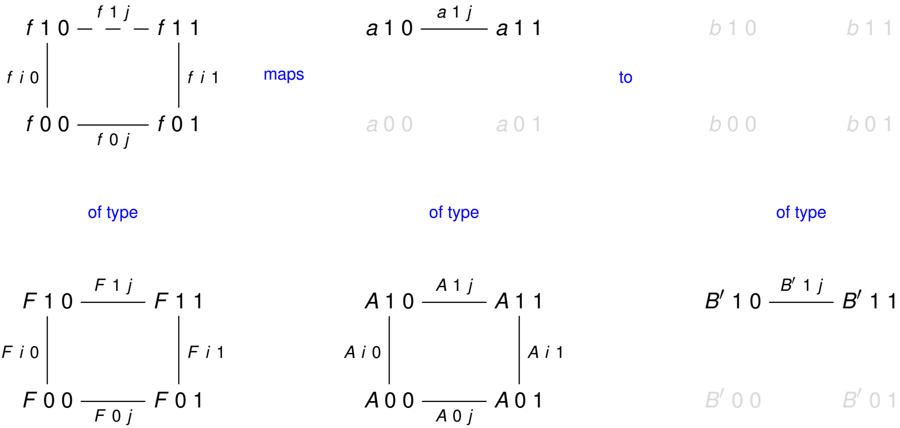


Kan fibrancy of Π

Bezem, Coquand & Huber (2014), Huber's Lic/PhD (2015/2016)

Abbreviate

- $b\ i\ j := f\ i\ j\ (a\ i\ j)$
- $F\ i\ j := (x : A\ i\ j) \to B\ i\ j\ x$
- $B'\ i\ j := B\ i\ j\ (a\ i\ j)$

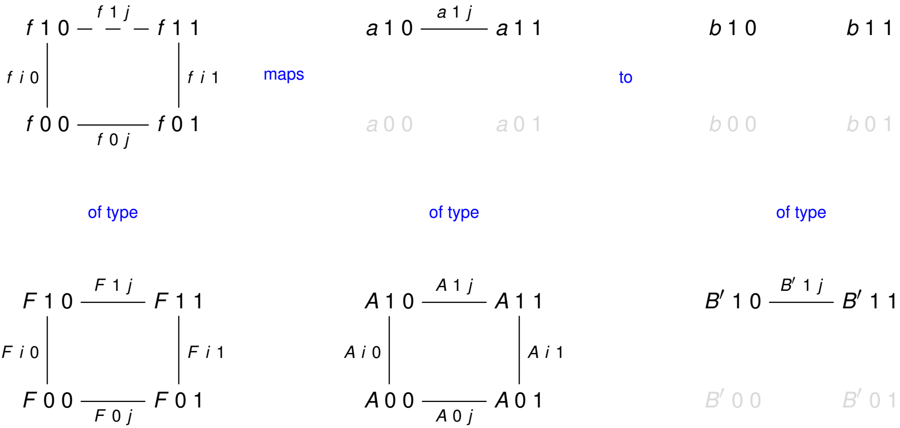


Kan fibrancy of Π

Bezem, Coquand & Huber (2014), Huber's Lic/PhD (2015/2016)

Abbreviate

- $b\ i\ j := f\ i\ j\ (a\ i\ j)$
- $F\ i\ j := (x : A\ i\ j) \to B\ i\ j\ x$
- $B'\ i\ j := B\ i\ j\ (a\ i\ j)$

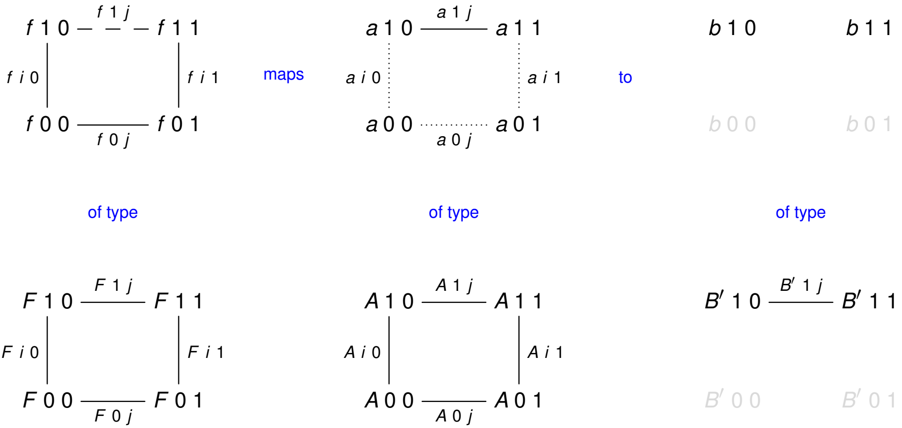


Kan fibrancy of Π

Bezem, Coquand & Huber (2014), Huber's Lic/PhD (2015/2016)

Abbreviate

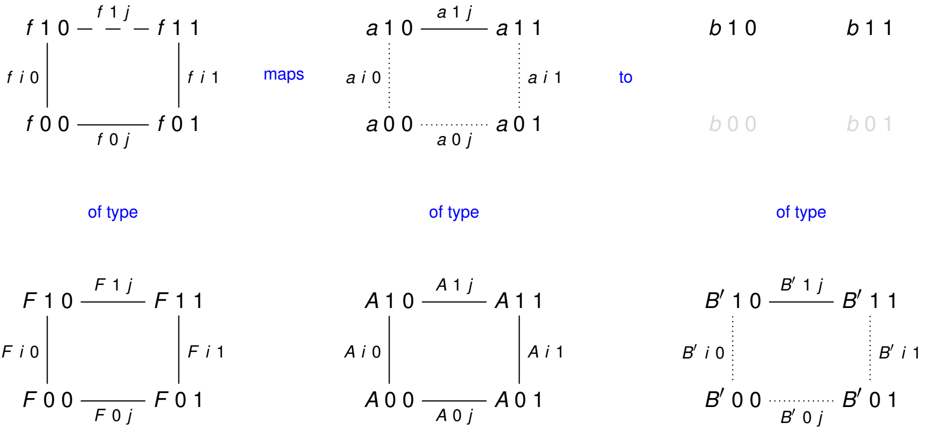
- $b\ i\ j := f\ i\ j\ (a\ i\ j)$
- $F\ i\ j := (x : A\ i\ j) \to B\ i\ j\ x$
- $B'\ i\ j := B\ i\ j\ (a\ i\ j)$



Bezem, Coquand & Huber (2014), Huber's Lic/PhD (2015/2016)

Abbreviate

- $b\ i\ j := f\ i\ j\ (a\ i\ j)$
- $F\ i\ j := (x : A\ i\ j) \to B\ i\ j\ x$
- $B'\ i\ j := B\ i\ j\ (a\ i\ j)$

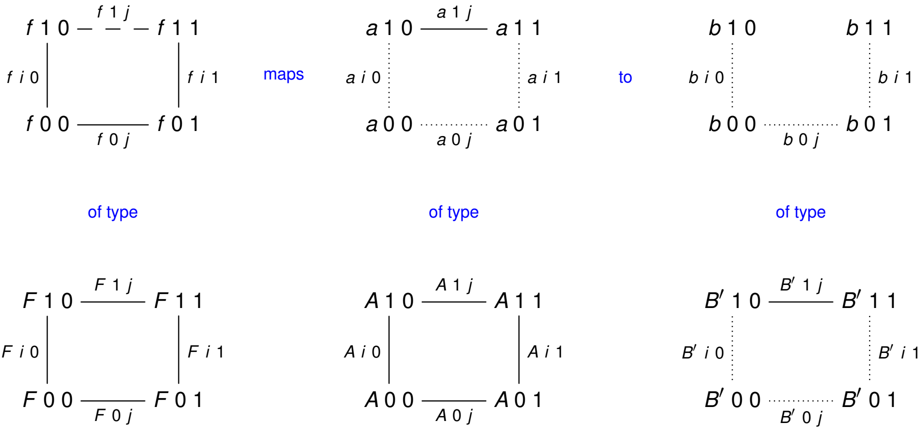


Kan fibrancy of Π

Bezem, Coquand & Huber (2014), Huber's Lic/PhD (2015/2016)

Abbreviate

- $b\ i\ j := f\ i\ j\ (a\ i\ j)$
- $F\ i\ j := (x : A\ i\ j) \to B\ i\ j\ x$
- $B'\ i\ j := B\ i\ j\ (a\ i\ j)$

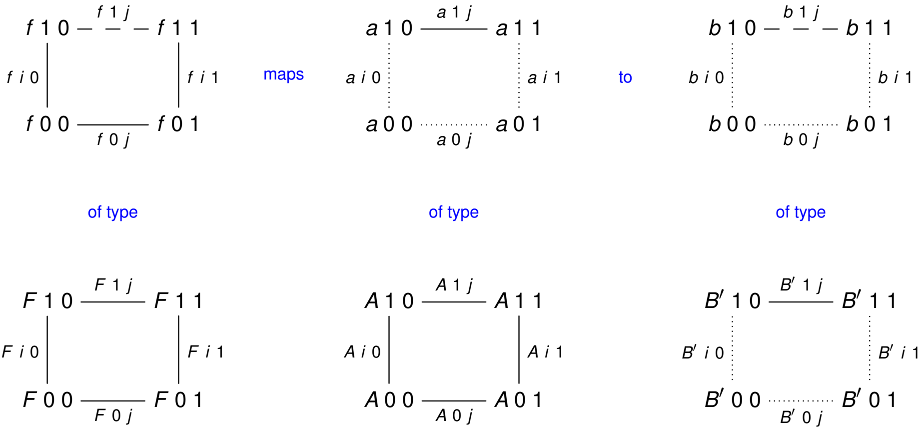


Kan fibrancy of Π

Bezem, Coquand & Huber (2014), Huber's Lic/PhD (2015/2016)

Abbreviate

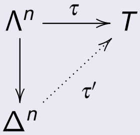
- $b\ i\ j := f\ i\ j\ (a\ i\ j)$
- $F\ i\ j := (x : A\ i\ j) \to B\ i\ j\ x$
- $B'\ i\ j := B\ i\ j\ (a\ i\ j)$



The Segal condition

Definition

$T \in$ **sSet** satisfies **Segal condition** if $\forall n, \tau. \exists ! \tau'$:



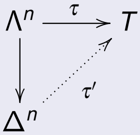
Then T is essentially a category.

Definition

$\Gamma \vdash T$ **type** is **Segal fibrant** if $\forall n, \gamma, \tau. \exists ! \tau'$:

Definition

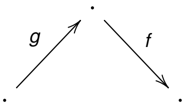
$T \in$ **sSet** satisfies **Segal condition** if $\forall n, \tau . \exists ! \tau'$:



Then T is essentially a category.

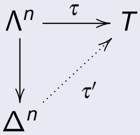
Definition

$\Gamma \vdash T$ **type** is **Segal fibrant** if $\forall n, \gamma, \tau . \exists ! \tau'$:



Definition

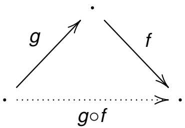
$T \in$ **sSet** satisfies **Segal condition** if $\forall n, \tau. \exists! \tau'$:



Then T is essentially a category.

Definition

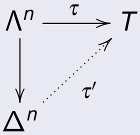
$\Gamma \vdash T$ **type** is **Segal fibrant** if $\forall n, \gamma, \tau. \exists! \tau'$:



The Segal condition

Definition

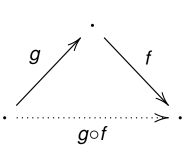
$T \in$ **sSet** satisfies **Segal condition** if $\forall n, \tau. \exists! \tau'$:

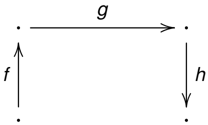


Then T is essentially a category.

Definition

$\Gamma \vdash T$ **type** is **Segal fibrant** if $\forall n, \gamma, \tau. \exists! \tau'$:

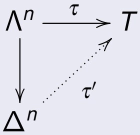




The Segal condition

Definition

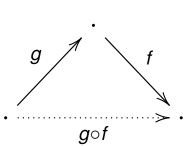
$T \in$ **sSet** satisfies **Segal condition** if $\forall n, \tau . \exists ! \tau'$:

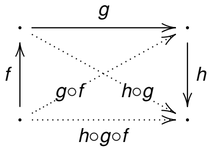


Then T is essentially a category.

Definition

$\Gamma \vdash T$ **type** is **Segal fibrant** if $\forall n, \gamma, \tau . \exists ! \tau'$:

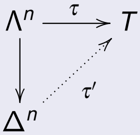




The Segal condition

Definition

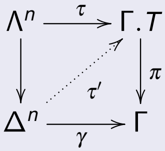
$T \in$ **sSet** satisfies **Segal condition** if $\forall n, \tau . \exists ! \tau'$:

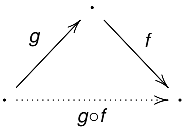


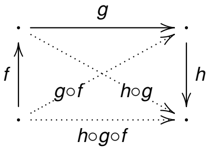
Then T is essentially a category.

Definition

$\Gamma \vdash T$ **type** is **Segal fibrant** if $\forall n, \gamma, \tau . \exists ! \tau'$:







Characterizing Segal types

If $\Gamma \vdash T$ **type** is Segal fibrant then:

- **Points** $x : \Delta^0 \to \Gamma$ map to **categories** $T[x]$,
- **Arrows** $\varphi : \Delta^1 \to \Gamma$ map to **pro-functors**[1] $T[\varphi] : T[x] \not\to T[y]$,
- **Triangles** $\Delta^2 \to \Gamma$ map to **pro-functor morphisms** $T[\chi] \circ T[\varphi] \Rightarrow T[\psi]$

- **Higher simplices** map to **commutative diagrams** of pro-functor morphisms.

[1] i.e. functors $T[x]^{\mathbf{op}} \times T[y] \to \mathbf{Set}$

Characterizing Segal types

If $\Gamma \vdash T$ **type** is Segal fibrant then:

- **Points** $x : \Delta^0 \to \Gamma$ map to **categories** $T[x]$,
- **Arrows** $\varphi : \Delta^1 \to \Gamma$ map to **pro-functors**[1] $T[\varphi] : T[x] \nrightarrow T[y]$,
- **Triangles** $\Delta^2 \to \Gamma$ map to **pro-functor morphisms** $T[\chi] \circ T[\varphi] \Rightarrow T[\psi]$

- **Higher simplices** map to **commutative diagrams** of pro-functor morphisms.

[1] i.e. functors $T[x]^{\mathbf{op}} \times T[y] \to \mathbf{Set}$

Characterizing Segal types

If $\Gamma \vdash T$ **type** is Segal fibrant then:

- **Points** $x : \Delta^0 \to \Gamma$ map to **categories** $T[x]$,
- **Arrows** $\varphi : \Delta^1 \to \Gamma$ map to **pro-functors**[1] $T[\varphi] : T[x] \nrightarrow T[y]$,
- **Triangles** $\Delta^2 \to \Gamma$ map to **pro-functor morphisms** $T[\chi] \circ T[\varphi] \Rightarrow T[\psi]$

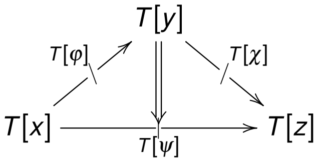
- **Higher simplices** map to **commutative diagrams** of pro-functor morphisms.

[1] i.e. functors $T[x]^{\mathbf{op}} \times T[y] \to \mathbf{Set}$

Characterizing Segal types

If $\Gamma \vdash T$ **type** is Segal fibrant then:

- **Points** $x : \Delta^0 \to \Gamma$ map to **categories** $T[x]$,
- **Arrows** $\varphi : \Delta^1 \to \Gamma$ map to **pro-functors**[1] $T[\varphi] : T[x] \nrightarrow T[y]$,
- **Triangles** $\Delta^2 \to \Gamma$ map to **pro-functor morphisms** $T[\chi] \circ T[\varphi] \Rightarrow T[\psi]$



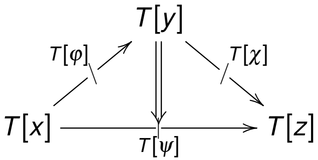
- **Higher simplices** map to **commutative diagrams** of pro-functor morphisms.

[1] i.e. functors $T[x]^{\mathbf{op}} \times T[y] \to \mathbf{Set}$

Characterizing Segal types

If $\Gamma \vdash T$ **type** is Segal fibrant then:

- **Points** $x : \Delta^0 \to \Gamma$ map to **categories** $T[x]$,
- **Arrows** $\varphi : \Delta^1 \to \Gamma$ map to **pro-functors**[1] $T[\varphi] : T[x] \nrightarrow T[y]$,
- **Triangles** $\Delta^2 \to \Gamma$ map to **pro-functor morphisms** $T[\chi] \circ T[\varphi] \Rightarrow T[\psi]$



- **Higher simplices** map to **commutative diagrams** of pro-functor morphisms.

[1] i.e. functors $T[x]^{\mathbf{op}} \times T[y] \to \mathbf{Set}$

Segal fibrancy of Π

Giraud (1964)

Abbreviate

- $b := f\ a$
- $F := (x : A) \to B\ x$
- $B' := B\ a$

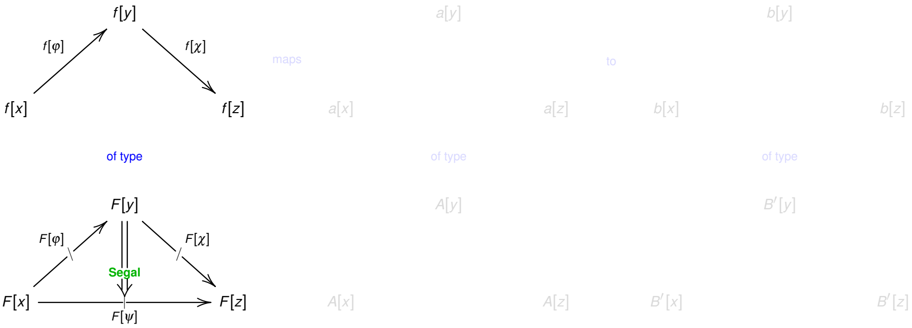
In non-dep. cat. theory: $a[y] := a[x], a[\varphi] = \mathrm{id}$.

Segal fibrancy of Π

Giraud (1964)

Abbreviate

- $b := f\ a$
- $F := (x : A) \to B\ x$
- $B' := B\ a$



In non-dep. cat. theory: $a[y] := a[x], a[\varphi] = \mathrm{id}$.

Segal fibrancy of Π

Giraud (1964)

Abbreviate

- $b := f\ a$
- $F := (x : A) \to B\ x$
- $B' := B\ a$

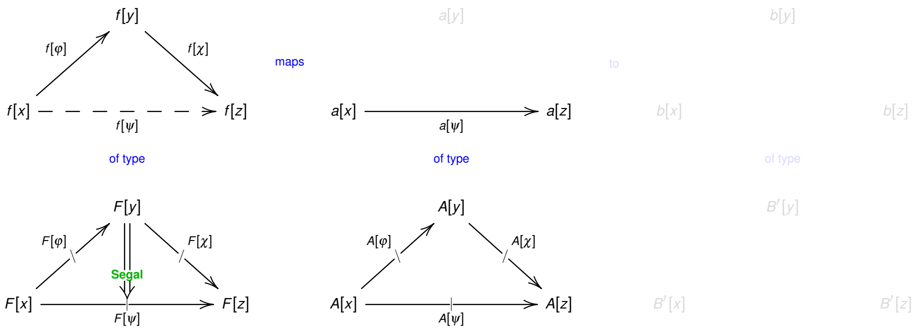
In non-dep. cat. theory: $a[y] := a[x], a[\varphi] = \mathrm{id}$.

Segal fibrancy of Π

Giraud (1964)

Abbreviate

- $b := f\ a$
- $F := (x : A) \to B\ x$
- $B' := B\ a$



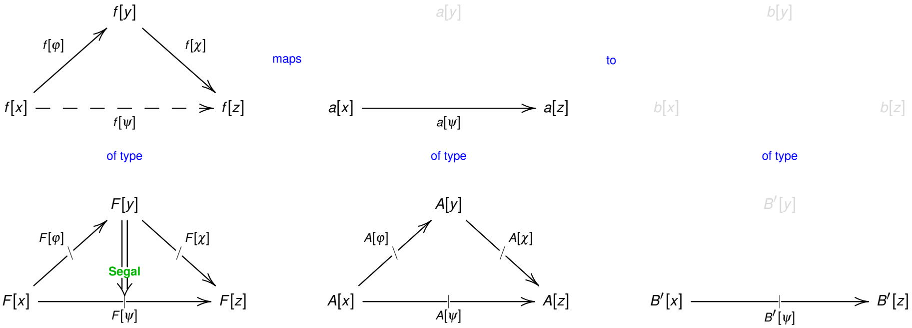
In non-dep. cat. theory: $a[y] := a[x], a[\varphi] = \mathrm{id}$.

Segal fibrancy of Π

Giraud (1964)

Abbreviate

- $b := f\ a$
- $F := (x : A) \to B\ x$
- $B' := B\ a$

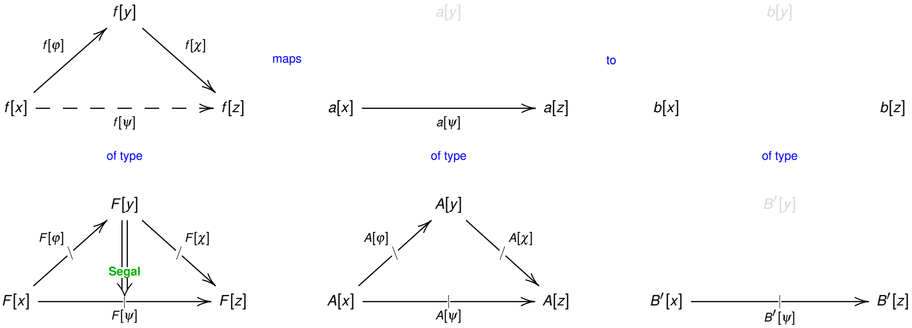


In non-dep. cat. theory: $a[y] := a[x], a[\varphi] = \mathrm{id}.$

Giraud (1964)

Abbreviate

- $b := f\ a$
- $F := (x : A) \to B\ x$
- $B' := B\ a$



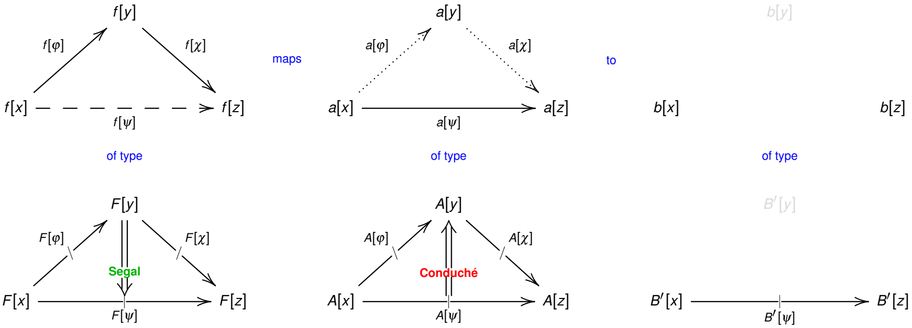
In non-dep. cat. theory: $a[y] := a[x], a[\varphi] = \mathrm{id}$.

Segal fibrancy of Π

Giraud (1964)

Abbreviate

- $b := f\ a$
- $F := (x : A) \to B\ x$
- $B' := B\ a$



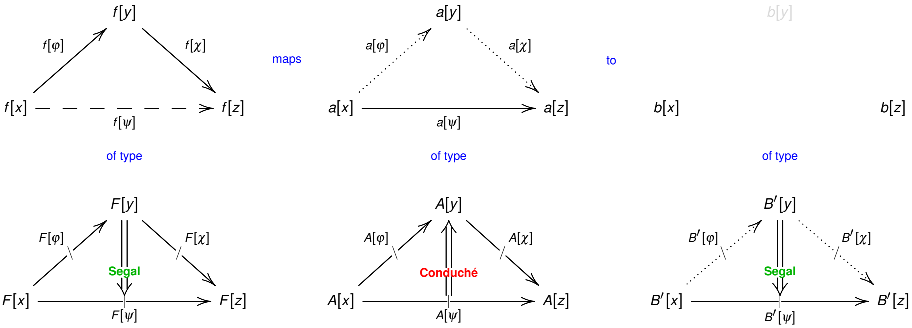
In non-dep. cat. theory: $a[y] := a[x], a[\varphi] = \mathrm{id}$.

Segal fibrancy of Π

Giraud (1964)

Abbreviate

- $b := f\ a$
- $F := (x : A) \to B\ x$
- $B' := B\ a$



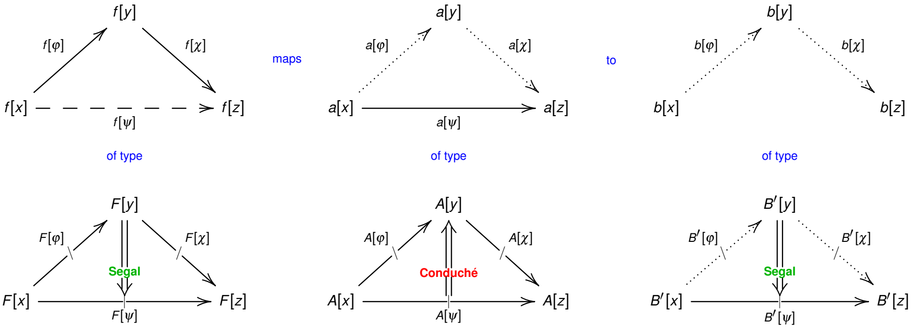
In non-dep. cat. theory: $a[y] := a[x], a[\varphi] = \mathrm{id}$.

Segal fibrancy of Π

Giraud (1964)

Abbreviate

- $b := f\ a$
- $F := (x : A) \to B\ x$
- $B' := B\ a$

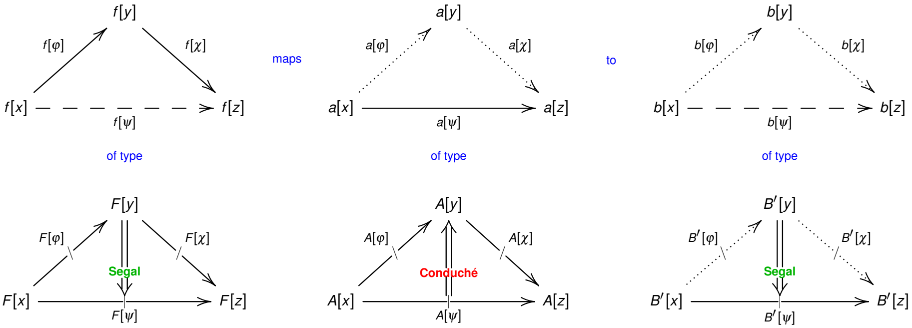


In non-dep. cat. theory: $a[y] := a[x], a[\varphi] = \mathrm{id}$.

Giraud (1964)

Abbreviate

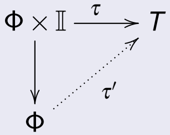
- $b := f\ a$
- $F := (x : A) \to B\ x$
- $B' := B\ a$



In non-dep. cat. theory: $a[y] := a[x], a[\varphi] = \mathrm{id}$.

Definition

$T \in$ **cSet** is **discrete** if
$\forall \Phi, \tau . \exists ! \tau'$:



Then T is essentially a **set**.

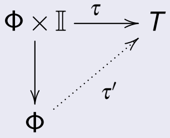
Definition

$\Gamma \vdash T$ **type** is **discrete** if
$\forall \Phi, \gamma, \tau . \exists ! \tau'$:

(Identity extension lemma)

Definition

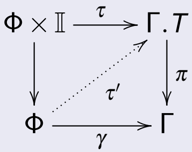
$T \in$ **cSet** is **discrete** if
$\forall \Phi, \tau . \exists! \tau'$:



Then T is essentially a **set**.

Definition

$\Gamma \vdash T$ **type** is **discrete** if
$\forall \Phi, \gamma, \tau . \exists! \tau'$:



(Identity extension lemma)

Reynolds (1983), Atkey, Ghani & Johann (2014)

Abbreviate

- $F := (x : A) \to B\ x$

$f\ 0 \overset{}{\underset{f\ i}{\longrightarrow}} f\ 1$ maps a — a to $f\ 0\ a$ — $f\ 1\ a$

of type — of type — of type

$F \underset{F}{\longrightarrow} F$ — A — A — $B\ a$ — $B\ a$

Note: We didn't use discreteness of A!

Reynolds (1983), Atkey, Ghani & Johann (2014)

Abbreviate

- $F := (x : A) \to B\ x$

$f\ 0 \underset{f\ i}{\text{———}} f\ 1$ maps a a to $f\ 0\ a$ $f\ 1\ a$

of type of type of type

$F \underset{F}{\text{———}} F$ A A $B\ a$ $B\ a$

Note: We didn't use discreteness of A!

Reynolds (1983), Atkey, Ghani & Johann (2014)

Abbreviate

- $F := (x : A) \to B\ x$

$f\ 0 \overset{}{\underset{f\ i}{\text{———}}} f\ 1$	maps	a	a	to	$f\ 0\ a$	$f\ 1\ a$
of type		of type			of type	
$F \underset{F}{\text{———}} F$		A	A		$B\ a$	$B\ a$

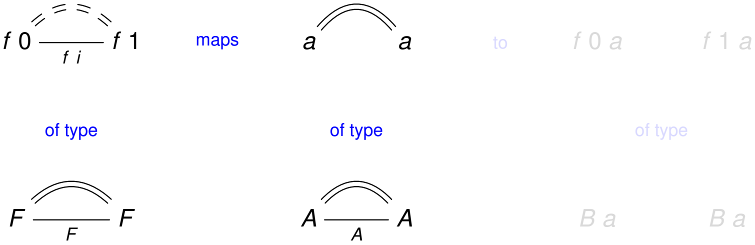
Note: We didn't use discreteness of A!

Discreteness of Π

Reynolds (1983), Atkey, Ghani & Johann (2014)

Abbreviate

- $F := (x : A) \to B\,x$

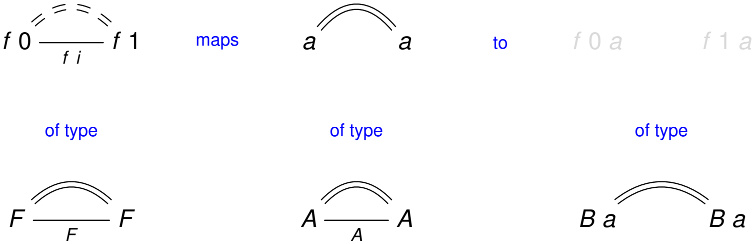


Note: We didn't use discreteness of A!

Reynolds (1983), Atkey, Ghani & Johann (2014)

Abbreviate

- $F := (x : A) \to B\ x$

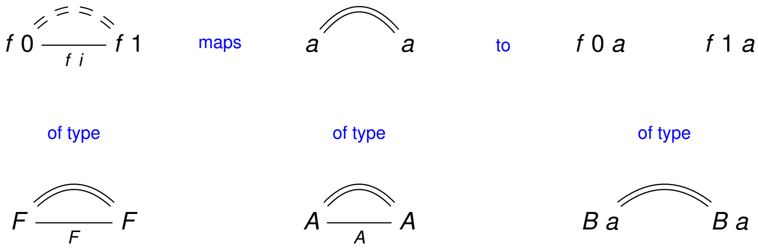


Note: We didn't use discreteness of A!

Reynolds (1983), Atkey, Ghani & Johann (2014)

Abbreviate

- $F := (x : A) \to B\ x$



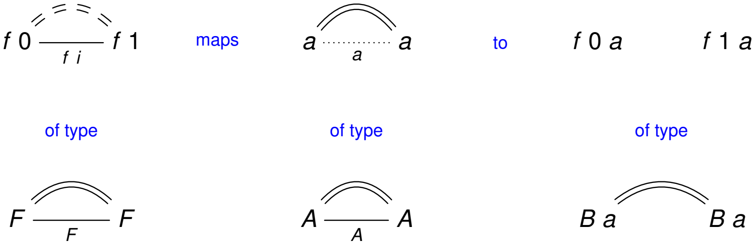
Note: We didn't use discreteness of *A*!

Discreteness of Π

Reynolds (1983), Atkey, Ghani & Johann (2014)

Abbreviate

- $F := (x : A) \to B\ x$

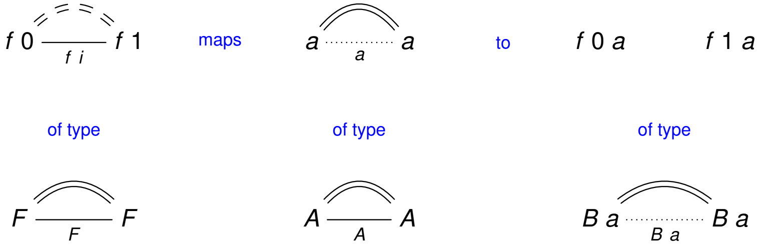


Note: We didn't use discreteness of A!

Reynolds (1983), Atkey, Ghani & Johann (2014)

Abbreviate

- $F := (x : A) \to B\,x$



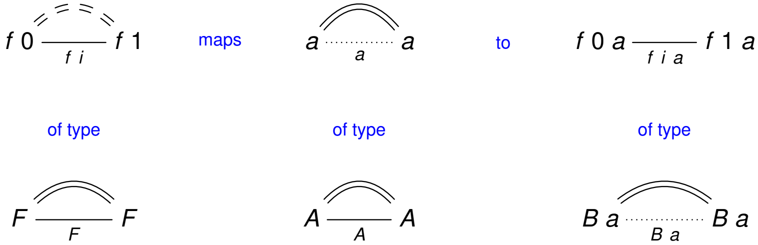
Note: We didn't use discreteness of A!

Discreteness of Π

Reynolds (1983), Atkey, Ghani & Johann (2014)

Abbreviate

- $F := (x : A) \to B\ x$

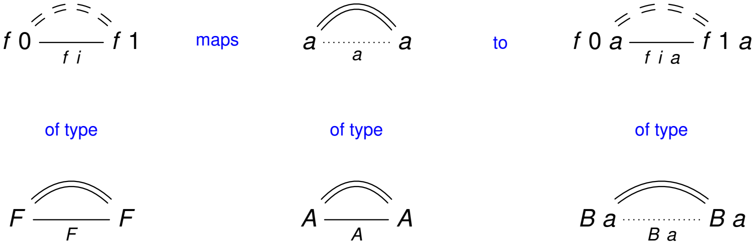


Note: We didn't use discreteness of A!

Reynolds (1983), Atkey, Ghani & Johann (2014)

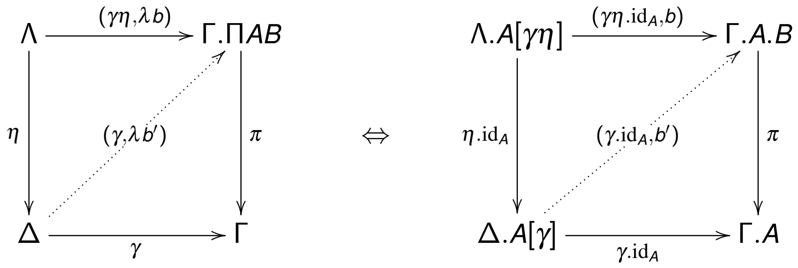
Abbreviate

- $F := (x : A) \to B\ x$



Note: We didn't use discreteness of *A*!

Let $\eta : \Lambda \to \Delta$ be a **generating left map**.



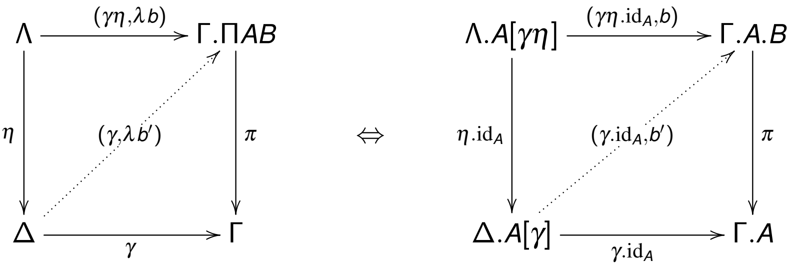
So if the **pullback** $\eta.\mathrm{id}_A$ of η is a **left map**, we're good!

Definition

Class of right maps is **robust** if generated by some left maps whose pullbacks are also left maps.

Fibrancy of Π in general

Let $\eta : \Lambda \to \Delta$ be a **generating left map**.



So if the **pullback** $\eta.\mathrm{id}_A$ of η is a **left map**, we're good!

Definition

Class of right maps is **robust** if generated by some left maps whose pullbacks are also left maps.

Theorem

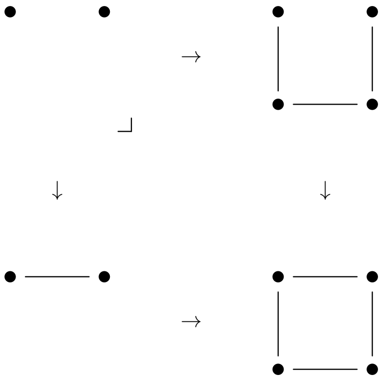
Discreteness is robust.

Proof:

$$\begin{array}{ccc} \Xi \times \mathbb{I} & \longrightarrow & \Phi \times \mathbb{I} \\ \big\downarrow \lrcorner & & \big\downarrow \\ \Xi & \longrightarrow & \Phi \end{array}$$

□

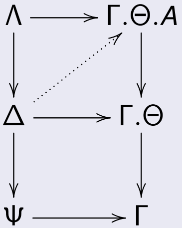
Force Kan fibrancy to be robust?



Then everything is equal!
(That's bad.)

Definition

$\Gamma | \Theta \vdash A\,\mathbf{fib}$ if:



for all gen. “**damped**” left maps.

Theorem

$$\frac{\Gamma \vdash A\,\mathbf{type} \qquad \Gamma.A | \Theta \vdash B\,\mathbf{fib}}{\Gamma | \Theta \vdash \Pi A B\,\mathbf{fib}}\ \textit{robust}$$

Definition

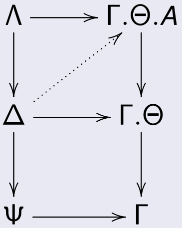
Contextual fibrancy is **robust** if generated by some ‘damped left maps’ whose pullbacks

are also damped left maps.

Contextual fibrancy

Definition

$\Gamma | \Theta \vdash A\,\mathbf{fib}$ if:



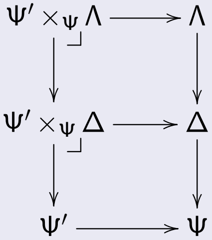
for all gen. “**damped**” left maps.

Theorem

$$\frac{\Gamma \vdash A\,\mathbf{type} \qquad \Gamma.A | \Theta \vdash B\,\mathbf{fib}}{\Gamma | \Theta \vdash \Pi A B\,\mathbf{fib}}\,\textit{robust}$$

Definition

Contextual fibrancy is **robust** if generated by some ‘damped left maps’ whose pullbacks

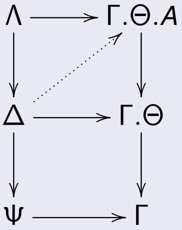


are also damped left maps.

Contextual fibrancy

Definition

$\Gamma | \Theta \vdash A\,\mathbf{fib}$ if:



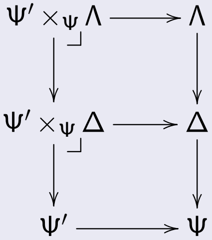
for all gen. “**damped**” left maps.

Theorem

$$\frac{\Gamma \vdash A\,\mathbf{type} \qquad \Gamma.A | \Theta \vdash B\,\mathbf{fib}}{\Gamma | \Theta \vdash \Pi A B\,\mathbf{fib}}\ \textit{robust}$$

Definition

Contextual fibrancy is **robust** if generated by some ‘damped left maps’ whose pullbacks



are also damped left maps.

Contextual **Kan** fibrancy [BT17]

$\sqcup \to \square \to -$

Contextual **Segal** fibrancy

$\Lambda^n \to \Delta^n \to \Delta^1$

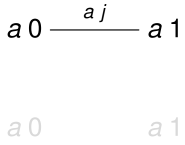
of type

of type

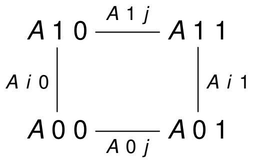
$\Gamma | \cdot \vdash T$ **fib**: Compose 1 heterog. line with multiple homog. lines.

Contextual **Kan** fibrancy [BT17]

$\sqcup \to \square \to -$



of type



Contextual **Segal** fibrancy

$\Lambda^n \to \Delta^n \to \Delta^1$

of type

$\Gamma|\cdot \vdash T$ **fib**: Compose 1 heterog. line with multiple homog. lines.

Contextual **Kan** fibrancy [BT17]

$\sqcup \to \square \to -$

$$a\,0 \xrightarrow{a\,j} a\,1$$

$$a\,0 \qquad a\,1$$

of type

$$\begin{array}{ccc} A\,0 & \overset{A\,j}{\text{———}} & A\,1 \\ \| & & \| \\ A\,0 & \underset{A\,j}{\text{———}} & A\,1 \end{array}$$

Contextual **Segal** fibrancy

$\Lambda^n \to \Delta^n \to \Delta^1$

$$a[z]$$

$$a[x] \xrightarrow[a[\psi]]{} a[z]$$

of type

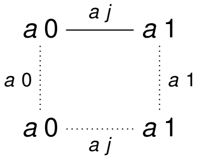
$$\begin{array}{ccccc} & & A[y] & & \\ & A[\varphi] \nearrow & \Uparrow & \searrow A[\chi] & \\ A[x] & & \xrightarrow[A[\psi]]{} & & A[z] \end{array}$$

Conduché

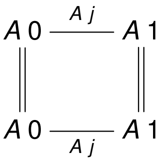
$\Gamma | \cdot \vdash T$ **fib**: Compose 1 heterog. line with multiple homog. lines.

Contextual **Kan** fibrancy [BT17]

$\sqcup \to \square \to -$



of type



Contextual **Segal** fibrancy

$\Lambda^n \to \Delta^n \to \Delta^1$

of type

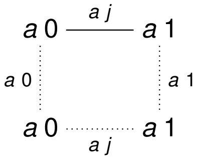
$\Gamma|\cdot \vdash T$ **fib**: Compose 1 heterog. line with multiple homog. lines.

Contextual **Kan** fibrancy [BT17]

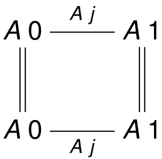
$\sqcup \to \square \to -$

Contextual **Segal** fibrancy

$\Lambda^n \to \Delta^n \to \Delta^1$



of type

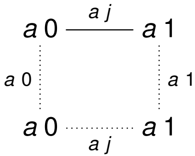


of type

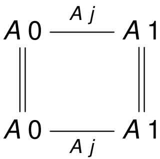
$\Gamma|\cdot \vdash T$ **fib**: Compose 1 heterog. line with multiple homog. lines.

Contextual **Kan** fibrancy [BT17]

$\sqcup \to \square \to -$

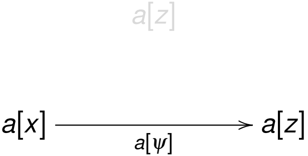


of type

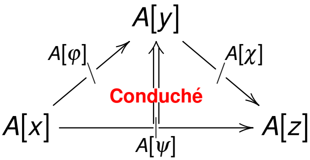


Contextual **Segal** fibrancy

$\Lambda^n \to \Delta^n \to \Delta^1$



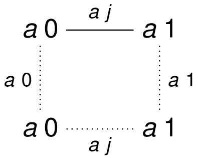
of type



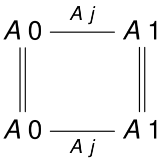
$\Gamma | \cdot \vdash T$ **fib**: Compose 1 heterog. line with multiple homog. lines.

Contextual **Kan** fibrancy [BT17]

$\sqcup \to \square \to -$

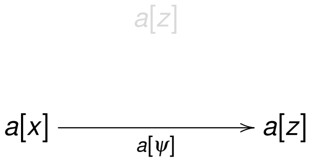


of type

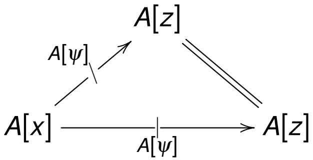


Contextual **Segal** fibrancy

$\Lambda^n \to \Delta^n \to \Delta^1$



of type



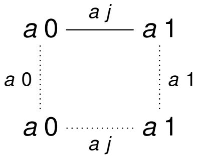
$\Gamma|\cdot \vdash T$ **fib**: Compose 1 heterog. line with multiple homog. lines.

Contextual **Kan** fibrancy [BT17]

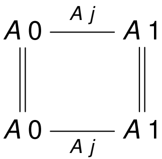
$$\sqcup \to \square \to -$$

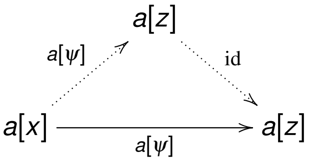
Contextual **Segal** fibrancy

$$\Lambda^n \to \Delta^n \to \Delta^1$$

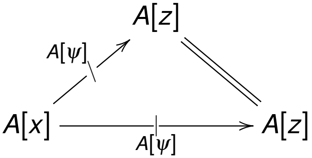


of type



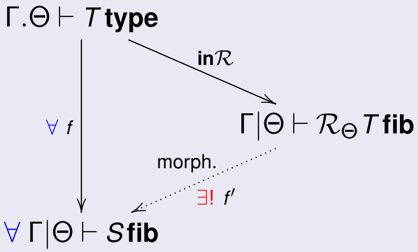


of type



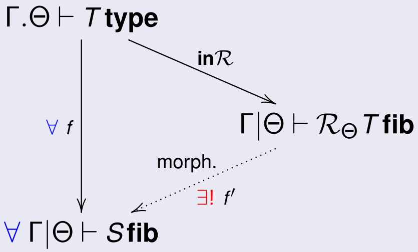
$\Gamma | \cdot \vdash T$ **fib**: Compose 1 heterog. line with multiple homog. lines.

Definition (Contextual fibrant replacement)



(Defined up to isomorphism.)

Definition (Contextual fibrant replacement)

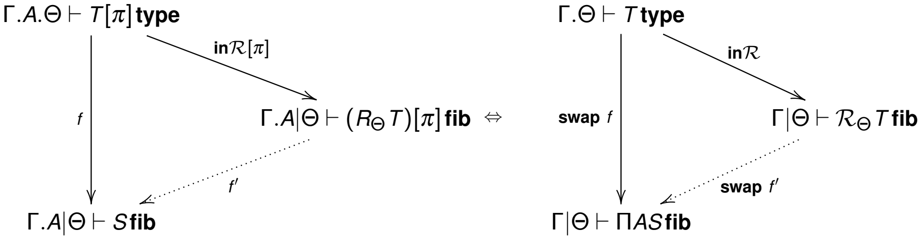


(Defined up to isomorphism.)

Theorem

Natural in Γ: $(\mathcal{R}_\Theta T)[\sigma] \cong \mathcal{R}_\Theta(T[\sigma])$.

Proof ($\sigma = \pi : \Gamma.A \to \Gamma$).



Take home message

Robustness:

- Makes $\Pi A B\,\mathbf{fib}$ if $B\,\mathbf{fib}$,
- Makes $\mathcal{R}$ natural,
- Is more achievable with contextual fibrancy.

Question

Is robustness "exactly" what can be internalized?

Thanks!

Related talk:
On HITs in Cubical TT
Coquand, Huber & Mörtberg
(Wednesday @ LICS)

Questions?

Take home message

Robustness:

- Makes $\Pi A B\,\mathbf{fib}$ if $B\,\mathbf{fib}$,
- Makes $\mathcal{R}$ natural,
- Is more achievable with contextual fibrancy.

Question

Is robustness “exactly” what can be internalized?

Thanks!

Related talk:
On HITs in Cubical TT
Coquand, Huber & Mörtberg
(Wednesday @ LICS)

Questions?

Take home message

Robustness:

- Makes $\Pi A B\,\mathbf{fib}$ if $B\,\mathbf{fib}$,
- Makes $\mathcal{R}$ natural,
- Is more achievable with contextual fibrancy.

Question

Is robustness “exactly” what can be internalized?

Thanks!

Related talk:
On HITs in Cubical TT
Coquand, Huber & Mörtberg
(Wednesday @ LICS)

Questions?

Take home message

Robustness:

- Makes $\Pi A B\,\mathbf{fib}$ if $B\,\mathbf{fib}$,
- Makes $\mathcal{R}$ natural,
- Is more achievable with contextual fibrancy.

Question

Is robustness “exactly” what can be internalized?

Thanks!

Related talk:
On HITs in Cubical TT
Coquand, Huber & Mörtberg
(Wednesday @ LICS)

Questions?

Take home message

Robustness:

- Makes $\Pi A B\,\mathbf{fib}$ if $B\,\mathbf{fib}$,
- Makes $\mathcal{R}$ natural,
- Is more achievable with contextual fibrancy.

Question

Is robustness “exactly” what can be internalized?

Thanks!

Related talk:
On HITs in Cubical TT
Coquand, Huber & Mörtberg
(Wednesday @ LICS)

Questions?