Dependable Atomicity in Type Theory

Andreas Nuyts[1], Dominique Devriese[2]

[1]KU Leuven, Belgium
[2]Vrije Universiteit Brussel, Belgium

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Presheaf semantics can model:

- **Parametricity** (preservation of **relations**),
- **HoTT** (preservation of **equivalences**),
- **Directed TT** (preservation of **homomorphisms**).

Operators for using the power of presheaves **within type theory**?

- Cubical HoTT: Glue,
- NVD17, ND18 (parametricity): Glue, Weld
  $\to$ seems to lack expressive power,
- Moulin16 (PhD on internal param'ty):
  Ψ (relativity), Φ (functional relativity)
  $\to$ requires substructural (affine-like) interval variables,
- LOPS18 (internal universes): $\surd$,
  $\to$ based on awkward postulates.

Conjecture

The following should suffice:

- **Dependable atomicity**,
- Orton and Pitts's **strictness axiom**.

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What's the Power of Presheaves

Theorem

Every presheaf category is a model of DTT...

...with extra tools!

Presheaf operators

Presheaf categories model:

- **Prop** (subobject classifier),
- Definitional **extension types** $A[\varphi \mathrel{?} a]$,
- Orton and Pitts's **strictness** axiom (rephrased):

$$\frac{\Gamma \vdash \varphi : \mathbf{Prop} \quad \Gamma, \varphi \vdash A : \mathcal{U} \quad \Gamma \vdash B : \mathcal{U} \quad \Gamma, \varphi \vdash i : A \cong B}{\Gamma \vdash A' : \mathcal{U}[\varphi \mathrel{?} A] \quad \Gamma \vdash i' : (A' \cong B)[\varphi \mathrel{?} i]}$$

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Definition

T is **atomic** if $T \to -$ has a right adjoint $\sqrt[T]{-}$.

Theorem

Yoneda-embeddings (e.g. $\mathbb{I}$) are atomic.

Example

If T is atomic, then

$$(T \to A \uplus B) \to (T \to A) \uplus (T \to B)$$

$$\cong$$

$$A \uplus B \to \sqrt[T]{(T \to A) \uplus (T \to B)}$$

$$\cong$$

$$\ldots \times \left(B \to \sqrt[T]{(T \to A) \uplus (T \to B)}\right)$$

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$$\ldots \times ((T \to B) \to (T \to A) \uplus (T \to B))$$

Corollary

Bool is not atomic.

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Definition

T is **dependably atomic** if
$(x : T) \to - : \mathbf{Ty}(\Gamma, x : T) \to \mathbf{Ty}(\Gamma)$
has a right adjoint **“transpension”**
$x \between - : \mathbf{Ty}(\Gamma) \to \mathbf{Ty}(\Gamma, x : T)$.

Theorem

Yoneda-embeddings (e.g. $\mathbb{I}$) are dependably atomic.

In cartesian settings

$$(\mathbb{I} \to \sqcup) = ((i : \mathbb{I}) \to \sqcup) \circ wkn$$
$$\dashv \sqrt[\mathbb{I}]{\sqcup} = (i : \mathbb{I}) \to (i \between \sqcup)$$

Typing rules
(Natural in Γ and Δ)

Formation:

$$\frac{\Gamma, (i : \mathbb{I}) \multimap \Delta\ i \vdash A\,\mathsf{type}}{\Gamma, i : \mathbb{I}, \Delta\ i \vdash i \between A\,\mathsf{type}}$$

Introduction:

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Elimination (simplified):

$$\mathbf{unmerid} : ((i : \mathbb{I}) \multimap i \between A) \to A.$$

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has a right adjoint **“transpension”**
$x \between - : \mathbf{Ty}(\Gamma) \to \mathbf{Ty}(\Gamma, x : T)$.

Theorem

Yoneda-embeddings (e.g. $\mathbb{I}$) are dependably atomic.

In cartesian settings

$$(\mathbb{I} \to \sqcup) \; = ((i : \mathbb{I}) \to \sqcup) \circ wkn$$
$$\dashv \sqrt[\mathbb{I}]{\sqcup} \; = (i : \mathbb{I}) \to (i \between \sqcup)$$

Typing rules

(Natural in Γ and Δ)

Formation:

$$\frac{\Gamma, (i : \mathbb{I}) \multimap \Delta\ i \vdash A\,\mathbf{type}}{\Gamma, i : \mathbb{I}, \Delta\ i \vdash i \between A\,\mathbf{type}}$$

Introduction:

$$\frac{\Gamma, (i : \mathbb{I}) \multimap \Delta\ i \vdash a : A\,\mathbf{type}}{\Gamma, i : \mathbb{I}, \Delta\ i \vdash \mathbf{merid}\ a\ i : i \between A\,\mathbf{type}}$$

Elimination (simplified):

$$\mathbf{unmerid} : ((i : \mathbb{I}) \multimap i \between A) \to A.$$

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Let's call it Transpension

Claim

The type $i \between A$ **looks a bit like** ($\not\cong$) the HIT with constructors:

- **north** : $0 \between A$,
- **south** : $1 \between A$,
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Moulin's Ψ (relativity) can be built from $- \between -$ and strictness.

merid from introduction rule.

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Rules invertible in the model
$\Rightarrow$ $0 \between A$ is a singleton.

Corresp. **dependent eliminator**:

- is not always sound for all motives,
- is probably equivalent to **Moulin's** Φ (func. relativity).

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Roadmap / Plan

- Settle on typing rules,
- Support Transpension type in Menkar,
- Formal claims about its power.

Conclusion

Looks like we can internalize presheaf semantics using:

- **Dependable atomicity** (transpension type),
- Orton and Pitts's **strictness axiom.**

Thanks!

Questions?