# Degrees of Relatedness

## A Unified Framework for Parametricity, Irrelevance, Ad Hoc Polymorphism, Intersections, Unions and Algebra in Dependent Type Theory

**Andreas Nuyts**, Dominique Devriese

KU Leuven, Belgium

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# Parametricity

$$if : \forall X.\mathrm{Bool} \to X \to X \to X$$

**Free Theorem**

If $f : A \to B$ then

$$f\ (if_A\ c\ a\ a') = if_B\ c\ (f\ a)\ (f\ a') : B$$

- System $F_\omega$, Haskell:
  **Type-level** args are parametric.
- DTT: Types can be values, values can be used at type-level.
  $\Rightarrow$ Explicit parametricity annotations (**par** as a **modality**) [1]

$$if : (\mathbf{par} \mid X : \mathcal{U}) \to \mathrm{Bool} \to X \to X \to X$$

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# Ad hoc polymorphism

Law of excluded middle (wrong):

$$lem : (\mathbf{par} \mid X : \mathcal{U}) \to X \uplus (X \to \text{Empty})$$

**Free Theorem** (contradiction!)

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Law of excluded middle (sound):

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Typecase (conceivably sound):

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**Irrelevance** $:=$ ignored by definitional equality

**Sized lists:**

- $nil_X : (\mathbf{irr} \mid n : \mathbb{N}) \to (\mathbf{irr} \mid 0 < n) \to \mathrm{List}_n\ X$,
- $cons_X : (\mathbf{irr} \mid m\ n : \mathbb{N}) \to (\mathbf{irr} \mid m < n) \to X \to \mathrm{List}_m\ X \to \mathrm{List}_n\ X$

Two ways to annotate $[a]$:

- $as_2 :\equiv cons_A\ 2\ 5\ \_\ a\ (nil_A\ 2\ \_) : \mathrm{List}_5\ A$, $nil_A\ 2\ \_ : \mathrm{List}_2\ A$
- $as_3 :\equiv cons_A\ 3\ 5\ \_\ a\ (nil_A\ 3\ \_) : \mathrm{List}_5\ A$, $nil_A\ 3\ \_ : \mathrm{List}_3\ A$
- $cons_A\ \bullet\ \bullet\ \_\ a\ (nil_A\ \bullet\ \_) : \mathrm{List}_5\ A \Rightarrow as_2 \equiv as_3$,

Irrelevance is a **dependent** generalization of **constancy**.

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Irrelevance is a **dependent** generalization of **constancy**.

That gives us **3 modalities**:

- **par** parametricity,
- **hoc** ad hoc polym.,
- **irr** irrelevance.

Can we handle that?

# Modalities interact with **type dependencies**

Given a modality $\mu$, in order to talk about

$$(\mu \mid x : A) \to B\,x,$$

what **modality cod** $\mu$ do we require for the dep. **codomain**

$$B : (\mathbf{cod}\ \mu \mid A) \to \mathcal{U}$$

?

# Codomain of **irrelevant** functions

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A point of dispute:

- **cod irr** = **irr**? [2] $\Rightarrow$ **Not usable for sized types.**
- **cod irr** = **hoc**? [3] i.e. $B$ can be anything?
  **Problem** with $\eta$-law. [4]
- Solution:
  **cod irr** = **shi** (**shape-irrelevance**, .. in Agda) [5]

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- **cod par** = **par**? i.e. $B : (\mathbf{par} \mid X : \mathcal{U}) \to \mathcal{U}$?

**Free Theorem**

$B$ is **constant**.

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**Free Theorem!?**

- Solution:
  **cod par** = **con** (**continuity**) [6]
  e.g. $\times : * \to * \to *$ in System $F_\omega$.

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# Modalities interact with **each other**

Given

$$f : (\mu \mid A) \to B,$$
$$g : (\nu \mid B) \to C,$$

what is the modality $\nu \circ \mu$ of

$$g \circ f : (\nu \circ \mu \mid A) \to C$$

?

$$\mathit{if}_{(\mathsf{List}_4\ A)} : \mathsf{Bool} \to \mathsf{List}_4\ A \to \mathsf{List}_4\ A \to \mathsf{List}_4\ A$$
$$\mathit{if}_{(\mathsf{List}_5\ A)} : \mathsf{Bool} \to \mathsf{List}_5\ A \to \mathsf{List}_5\ A \to \mathsf{List}_5\ A$$

- We can ignore irrelevant parts.
- *if* uses first arg. parametrically.
- List uses size index shape-irrelevantly.

So **par** $\circ$ **shi** $=$ **irr**?

# Modalities interact with the **type level**

For example:

**Theorem**

All continuous (non-ad-hoc) functions with **small** codomain are **parametric**.

*Takeuti (2001)*
*Krishnaswami & Dreyer (2013)*
*Atkey, Ghani & Johann (2014)*

Hence, System $F_\omega$ has no use for $(\textbf{con} \shortmid X : *) \to T(X)$.

We now have **5 modalities**:

| | |
|---|---|
| **par** | parametricity, |
| **con** | continuity, |
| **irr** | irrelevance (. in Agda), |
| **shi** | shape-irrelevance (.. in Agda), |
| **hoc** | ad hoc polym. |

These **interact**:

- with **type dependencies**,
- with **each other**,
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**Meaning? Rules? Soundness?**

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# Degrees of Relatedness

Equip types with **multiple, proof-relevant relations**:

- Just **equality** for **small types** (*Bool*, $\mathbb{N} \to \mathbb{N}$, ...),
- **More** for **larger** types ($\mathcal{U}_0 \to \mathcal{U}_0$, Grp, ...).

(We can decouple level and "depth": $\mathcal{U}_\ell^d : \mathcal{U}_{\ell+1}^{d+1}$.)

Modality describes how functions act on relations.

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| | **Value-level** objects<br>$a : A : \mathcal{U}_0$ can be | **Type-level** objects<br>$A : \kappa : \mathcal{U}_1$ can be |
|---|---|---|
| 0-related (het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$<br>$\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool} \to R \to R \to R} \mathit{if}_Y$ | $((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$<br>$([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$<br>$\ldots$ |
| 1-related | n/a | $\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\mathrm{List}_\bullet\ A : \mathrm{List}_4\ A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\ A$ |

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$$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$$

because

$$2+5 \equiv 7$$

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|---|---|---|
| 0-related (het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$<br>$\forall R.if_X \frown_0^{\mathrm{Bool} \to R \to R \to R} if_Y$ | $((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$<br>$([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$<br>$\dots$ |
| 1-related | n/a | $\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\mathrm{List}_\bullet\ A : \mathrm{List}_4\ A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\ A$ |

$$((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$$

because

$$(\lambda X.X)\ \mathrm{Bool} \equiv \mathrm{Bool}$$

| | **Value-level** objects $a : A : \mathcal{U}_0$ can be | **Type-level** objects $A : \kappa : \mathcal{U}_1$ can be |
|---|---|---|
| 0-related (het. eq.) | $(2 + 5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^R (\mathsf{true} : \mathsf{Bool})$<br>$\forall R.if_X \frown_0^{\mathsf{Bool} \to R \to R \to R} if_Y$ | $((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$<br>$([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$<br>... |
| 1-related | n/a | $\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$ |

$$([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$$

where

$$\mathsf{List}_\bullet\ A \in \mathrm{Rel}(\mathsf{List}_4\ A, \mathsf{List}_6\ A)$$

| | **Value-level** objects $a : A : \mathcal{U}_0$ can be | **Type-level** objects $A : \kappa : \mathcal{U}_1$ can be |
|---|---|---|
| 0-related (het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$<br>$\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$ | $((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$<br>$([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$<br>$\dots$ |
| 1-related | n/a | $\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$ |

$$([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$$

| | **Value-level** objects $a : A : \mathcal{U}_0$ can be | **Type-level** objects $A : \kappa : \mathcal{U}_1$ can be |
|---|---|---|
| 0-related (het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$<br>$\forall R.if_X \frown_0^{\mathrm{Bool} \to R \to R \to R} if_Y$ | $((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$<br>$([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$<br>... |
| 1-related | n/a | $\big((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\big) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\mathrm{List}_\bullet\ A : \mathrm{List}_4\ A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\ A$ |

$$(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$$
for some
$$R \in \mathrm{Rel}(\mathbb{N}, \mathrm{Bool})$$

| | **Value-level** objects $a : A : \mathcal{U}_0$ can be | **Type-level** objects $A : \kappa : \mathcal{U}_1$ can be |
|---|---|---|
| 0-related (het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ <br> $([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$ <br> $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$ <br> $\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool} \to R \to R \to R} \mathit{if}_Y$ | $((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$ <br> $([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$ <br> ... |
| 1-related | n/a | $\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ <br> $\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ <br> $\mathrm{List}_\bullet\ A : \mathrm{List}_4\ A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\ A$ |

$$(\mathit{if}_X : \mathrm{Bool} \to X \to X \to X) \frown_0^{\mathrm{Bool} \to R \to R \to R} (\mathit{if}_Y : \mathrm{Bool} \to Y \to Y \to Y)$$

for all

$$R \in \mathrm{Rel}(X, Y)$$

| | **Value-level** objects $a : A : \mathcal{U}_0$ can be | **Type-level** objects $A : \kappa : \mathcal{U}_1$ can be |
|---|---|---|
| 0-related (het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$<br>$\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool} \to R \to R \to R} \mathit{if}_Y$ | $((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$<br>$([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$<br>$\ldots$ |
| 1-related | n/a | $\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\mathrm{List}_\bullet\ A : \mathrm{List}_4\ A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\ A$ |

$(a : A) \frown_i^{R} (b : B)$ is always w.r.t. $R : (A : \mathcal{U}_n) \frown_{i+1}^{\mathcal{U}_n} (B : \mathcal{U}_n)$

$$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$$

| | **Value-level** objects $a : A : \mathcal{U}_0$ can be | **Type-level** objects $A : \kappa : \mathcal{U}_1$ can be |
|---|---|---|
| 0-related (het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$<br>$\forall R.if_X \frown_0^{\mathrm{Bool} \to R \to R \to R} if_Y$ | $((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$<br>$([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$<br>$\ldots$ |
| 1-related | n/a | $\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\mathrm{List}_\bullet\ A : \mathrm{List}_4\ A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\ A$ |

$$\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$$

| | **Value-level** objects $a : A : \mathcal{U}_0$ can be | **Type-level** objects $A : \kappa : \mathcal{U}_1$ can be |
|---|---|---|
| 0-related (het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \text{List}_4\ A) \frown_0^{\text{List}_\bullet\ A} ([] : \text{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\text{true} : \text{Bool})$<br>$\forall R.\mathit{if}_X \frown_0^{\text{Bool} \to R \to R \to R} \mathit{if}_Y$ | $((\lambda X.X)\ \text{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\text{Bool} : \mathcal{U}_0)$<br>$([] : \text{List}_4\ \kappa) \frown_0^{\text{List}_\bullet\ \kappa} ([] : \text{List}_6\ \kappa)$<br>$\dots$ |
| 1-related | n/a | $\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \text{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\text{List}_\bullet\ A : \text{List}_4\ A \frown_1^{\mathcal{U}_0} \text{List}_6\ A$ |

$$\text{List}_\bullet\ A : (\text{List}_4\ A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\text{List}_6\ A : \mathcal{U}_0)$$

| | **Value-level** objects $a : A : \mathcal{U}_0$ can be | **Type-level** objects $A : \kappa : \mathcal{U}_1$ can be |
|---|---|---|
| 0-related (het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$<br>$\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool} \to R \to R \to R} \mathit{if}_Y$ | $((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$<br>$([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$<br>$\ldots$ |
| 1-related | n/a | $\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\mathrm{List}_\bullet\ A : \mathrm{List}_4\ A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\ A$ |

See paper for $\frown_2$ (as of **kind-level**) and higher.

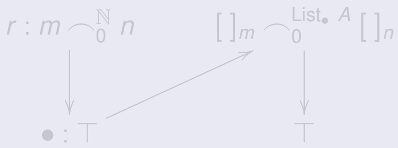
Four laws:

- **Reflexivity:** $(a : A) \frown^{A}_{i} (a : A)$
- **Weakening:** $((a : A) \frown^{R}_{i} (b : B)) \to ((a : A) \frown^{R}_{i+1} (b : B))$
- **Dependency:** $(a : A) \frown^{R}_{i} (b : B)$ presumes $R : A \frown^{\mathcal{U}_n}_{i+1} B$
- **Identity extension:** $(a : A) \frown^{A}_{0} (b : A)$ means $a \equiv b : A$.

# Understanding modalities (1)

**irr** : values → values

$$[\,] : (\mathbf{irr} \mid n : \mathbb{N}) \to \mathrm{List}_n\, A$$



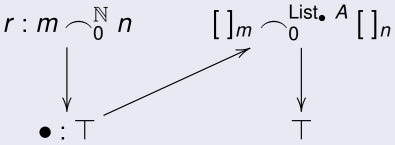

**shi** : values → types

$$\lambda n.\mathrm{List}_n\, A : (\mathbf{shi} \mid n : \mathbb{N}) \to \mathcal{U}_0$$

# Understanding modalities (1)

### **irr** : values → values

$$[\,] : (\mathbf{irr} \mid n : \mathbb{N}) \to \mathsf{List}_n\, A$$

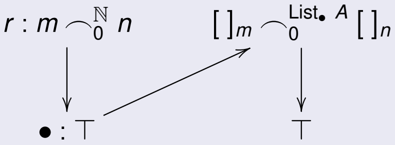



### **shi** : values → types

$$\lambda n.\mathsf{List}_n\, A : (\mathbf{shi} \mid n : \mathbb{N}) \to \mathcal{U}_0$$

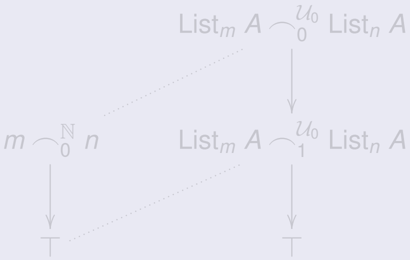
## **irr** : values $\to$ values

$$[\,] : (\mathbf{irr} \mid n : \mathbb{N}) \to \mathrm{List}_n\ A$$



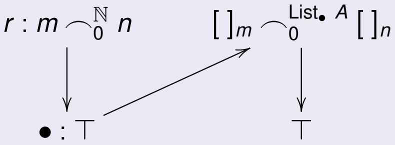

## **shi** : values $\to$ types

$$\lambda n.\mathrm{List}_n\ A : (\mathbf{shi} \mid n : \mathbb{N}) \to \mathcal{U}_0$$



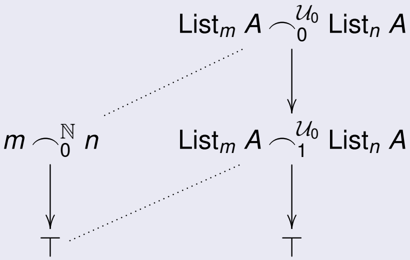
## irr : values → values

$$[\,] : (\mathbf{irr} \mid n : \mathbb{N}) \to \mathrm{List}_n\ A$$




## shi : values → types

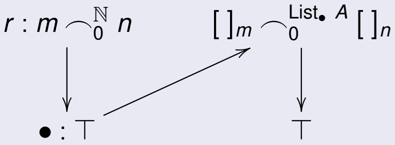
$$\lambda n.\mathrm{List}_n\ A : (\mathbf{shi} \mid n : \mathbb{N}) \to \mathcal{U}_0$$



# Understanding modalities (1)

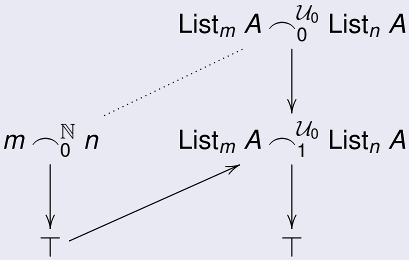
## **irr** : values → values

$$[\,] : (\mathbf{irr} \mid n : \mathbb{N}) \to \mathrm{List}_n\ A$$




## **shi** : values → types

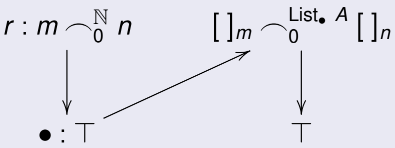
$$\lambda n.\mathrm{List}_n\ A : (\mathbf{shi} \mid n : \mathbb{N}) \to \mathcal{U}_0$$



# Understanding modalities (1)

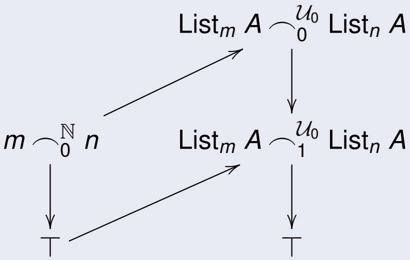
**irr** : values → values

$$[\,] : (\mathbf{irr} \mid n : \mathbb{N}) \to \mathrm{List}_n\ A$$




**shi** : values → types

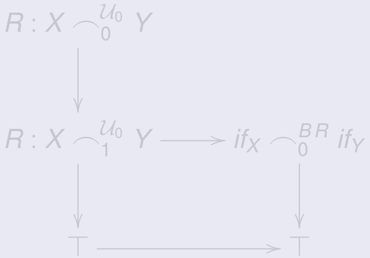
$$\lambda n.\mathrm{List}_n\ A : (\mathbf{shi} \mid n : \mathbb{N}) \to \mathcal{U}_0$$



# Understanding modalities (2)

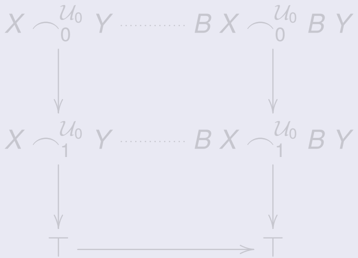
**par** : types $\rightarrow$ values

$$if : (\mathbf{par} \mid X : \mathcal{U}_0) \rightarrow B\ X$$



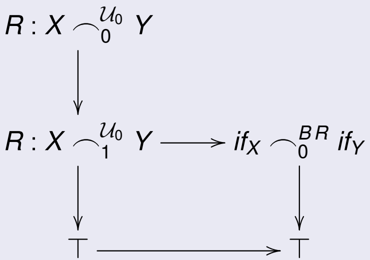

**con** : types $\rightarrow$ types

$$B : \mathcal{U}_0 \rightarrow \mathcal{U}_0$$
$$B\ X = \mathrm{Bool} \rightarrow X \rightarrow X \rightarrow X$$



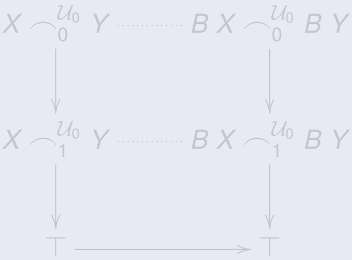
## **par** : types $\to$ values

$$\mathit{if} : (\mathbf{par} \mid X : \mathcal{U}_0) \to B\,X$$



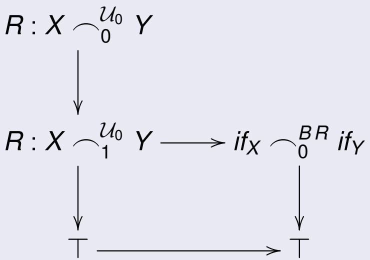

## **con** : types $\to$ types

$$B : \mathcal{U}_0 \to \mathcal{U}_0$$
$$B\,X = \mathrm{Bool} \to X \to X \to X$$



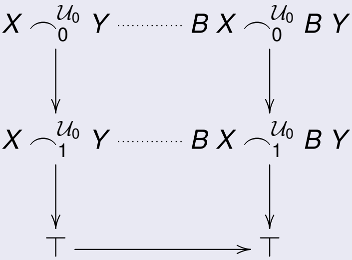
## **par** : types $\to$ values

$$if : (\mathbf{par} \mid X : \mathcal{U}_0) \to B\ X$$



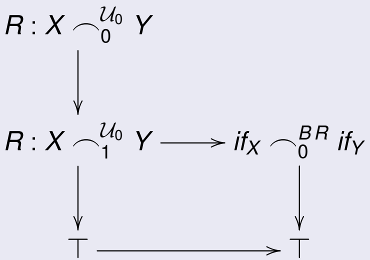

## **con** : types $\to$ types

$$B : \mathcal{U}_0 \to \mathcal{U}_0$$
$$B\ X = \mathrm{Bool} \to X \to X \to X$$



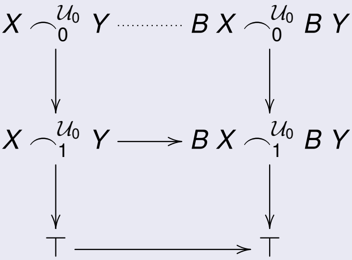
### **par** : types → values

$$\mathit{if} : (\mathbf{par} \mid X : \mathcal{U}_0) \to B\, X$$




### **con** : types → types

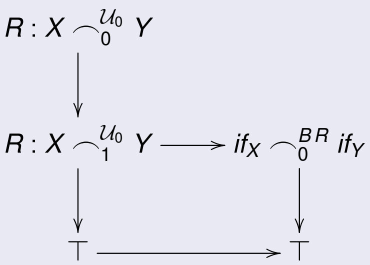
$$B : \mathcal{U}_0 \to \mathcal{U}_0$$
$$B\, X = \mathrm{Bool} \to X \to X \to X$$



# Understanding modalities (2)

**par** : types → values

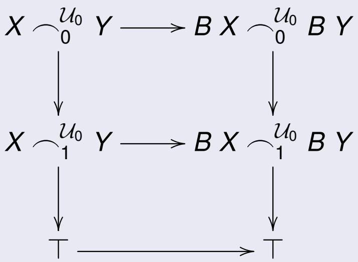
$$\mathit{if} : (\mathbf{par} \mid X : \mathcal{U}_0) \to B\,X$$




**con** : types → types

$$B : \mathcal{U}_0 \to \mathcal{U}_0$$

$$B\,X = \mathrm{Bool} \to X \to X \to X$$



# All modalities at lowest levels

| $(\mu \mid A) \to B$ | $B : \mathcal{U}_0$<br>values | $B : \mathcal{U}_1$<br>types | $B : \mathcal{U}_n$ |
|---|---|---|---|
| $A : \mathcal{U}_0$<br>values | **hoc**, **irr** | **hoc**, **shi**, **irr** | |
| $A : \mathcal{U}_1$<br>types | **hoc**, **par**, **irr** | **hoc**, **con**, **shi**,<br>**par**, **shi&par**, **irr** | |
| $A : \mathcal{U}_m$ | | | $\frac{(m+n+2)!}{(m+1)!(n+1)!}$ |

$if_{(\mathrm{List}_n\ A)}$

Irrelevant in $n$?

Yes if **par** $\circ$ **shi** $=$ **irr** $\surd$

$if_{(\mathsf{List}_n\ A)}$

Irrelevant in $n$?

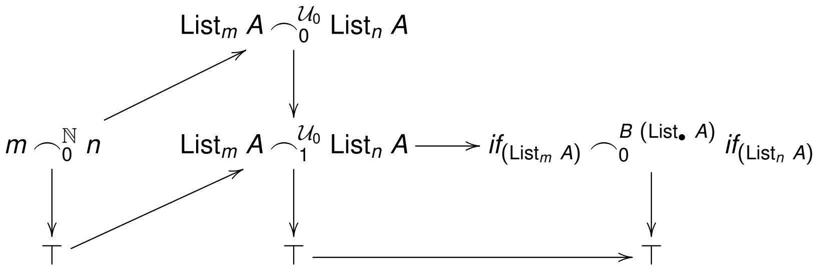
Yes if **par** $\circ$ **shi** $=$ **irr** $\surd$

# Composition of modalities

$if_{(\mathsf{List}_n\ A)}$

Irrelevant in $n$?

Yes if $\mathbf{par} \circ \mathbf{shi} = \mathbf{irr}$ $\surd$




**shi** **irr** **par**

# Composition of modalities

$if_{(\mathrm{List}_n\ A)}$

Irrelevant in $n$?

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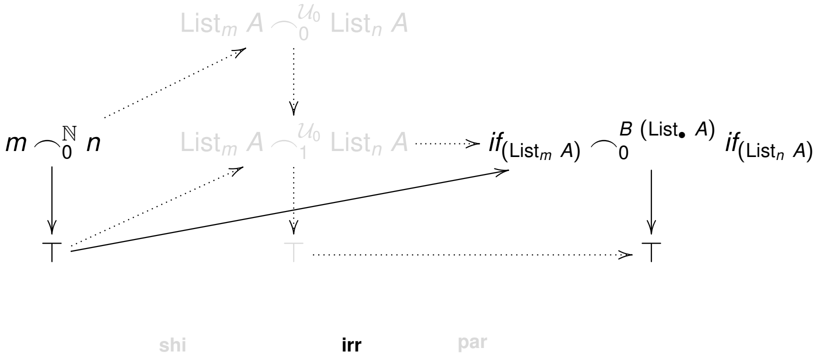

# Conclusion

- Unified framework (type system + presheaf model) for:
  - **parametricity**
  - **continuity**
  - **. irrelevance**
  - **.. shape-irrelevance**
  - **ad hoc polym.**
- Understanding of **interactions** with
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  - **type dependencies**,
  - **type level/depth** [7]
- Type-checking time **erasure of irrelevant subterms**
- Sheds light on: **algebra, unions, intersections, Prop, . . .**

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Licata & Shulman (2016), Licata, Shulman & Riley (2017)

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## Take home message

Describe function behaviour as action on degree of relatedness.
**par**, **con**, **irr**, **shi**, **hoc** are instances of this.

**Thanks!**

**Questions?**

**System $F_\omega$:**

**Free Theorem**

$$\forall X.(X \to A) \to (X \to B) \quad \cong \quad A \to B$$

**Dependent types:**

$leak : (X : \mathcal{U}) \to (X \to A) \to (X \to \mathcal{U})$

$leak\ X\ f\ x = X$

**Our solution:**

$(\textbf{par} \mid X : \mathcal{U}) \to (X \to A) \to (X \to \mathcal{U})$

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# Comparison with HoTT

| Degrees of Relatedness | HoTT |
|---|---|
| functions **act** on $\frown_i$ | functions **preserve** $\simeq$ |
| equality as $\frown_0$ | equality as $\simeq$ |
| relational HITs[8] | groupoidal HITs |
| depth: $\mathcal{U}^d_\ell : \mathcal{U}^{d+1}_{\ell+1}$ | $h$-level: $\mathcal{U}^h_\ell : \mathcal{U}^{h+1}_{\ell+1}$ |

[8] future work

| | **Value-level** objects<br>$a : A : \mathcal{U}_0$ can be | **Type-level** objects<br>$A : \kappa : \mathcal{U}_1$ can be | **Kind-level** objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be |
|---|---|---|---|
| 0-related<br>(het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$<br>$\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool}\to R\to R\to R} \mathit{if}_Y$ | $((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$<br>$([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$<br>... | $((\lambda\xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$<br>... |
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| 2-related | n/a | n/a | $V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$ |

| | **Value-level** objects<br>$a : A : \mathcal{U}_0$ can be | **Type-level** objects<br>$A : \kappa : \mathcal{U}_1$ can be | **Kind-level** objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be |
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$$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$$

because

$$2+5 \equiv 7$$

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$$([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$$

where

$$\mathrm{List}_\bullet\ A \in \mathrm{Rel}(\mathrm{List}_4\ A, \mathrm{List}_6\ A)$$

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$$(5 : \mathbb{N}) \frown_0^{R} (\text{true} : \text{Bool})$$

for some

$$R \in \mathrm{Rel}(\mathbb{N}, \text{Bool})$$

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$$(\mathit{if}_X : \mathsf{Bool} \to X \to X \to X) \frown_0^{\mathsf{Bool} \to R \to R \to R} (\mathit{if}_Y : \mathsf{Bool} \to Y \to Y \to Y)$$

for all

$$R \in \mathrm{Rel}(X, Y)$$

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| 2-related | n/a | n/a | $V : (\mathrm{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathrm{Mon} : \mathcal{U}_1)$ |

$$((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$$

because

$$(\lambda X.X)\ \mathrm{Bool} \equiv \mathrm{Bool}$$

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|---|---|---|---|
| 0-related<br>(het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$<br>$\forall R.if_X \frown_0^{\mathsf{Bool} \to R \to R \to R} if_Y$ | $((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$<br>$([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$<br>... | $((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$<br>... |
| 1-related | n/a | $\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$<br>$R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$<br>$R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$ | $\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$<br>$\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$ |
| 2-related | n/a | n/a | $V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$ |

$$([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$$

| | **Value-level** objects<br>$a : A : \mathcal{U}_0$ can be | **Type-level** objects<br>$A : \kappa : \mathcal{U}_1$ can be | **Kind-level** objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be |
|---|---|---|---|
| 0-related<br>(het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$<br>$\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool}\to R\to R\to R} \mathit{if}_Y$ | $((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$<br>$([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$<br>… | $((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \mathrm{List}_4\ \mathcal{A}) \frown_0^{\mathrm{List}_\bullet\ \mathcal{A}} ([] : \mathrm{List}_6\ \mathcal{A})$<br>… |
| 1-related | n/a | $\big((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\big) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\mathrm{List}_\bullet\ A : \mathrm{List}_4\ A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\ A$<br>$R : (G : \mathrm{Grp}) \frown_1^{\mathrm{Grp}} (H : \mathrm{Grp})$<br>$R : (G : \mathrm{Grp}) \frown_1^{V} (M : \mathrm{Mon})$ | $\big((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\big) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet\to\bullet\}}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\mathrm{List}_\bullet\ \kappa : \mathrm{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathrm{List}_6\ \kappa$<br>$\rho : (\alpha : \mathrm{Cat}) \frown_1^{\mathrm{Cat}} (\beta : \mathrm{Cat})$ |
| 2-related | n/a | n/a | $V : (\mathrm{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathrm{Mon} : \mathcal{U}_1)$ |

$(a : A) \frown_i^{R} (b : B)$ is always w.r.t. $R : (A : \mathcal{U}_n) \frown_{i+1}^{\mathcal{U}_n} (B : \mathcal{U}_n)$

$$\big((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\big) := \mathrm{Rel}(A, B)$$

| | **Value-level** objects<br>$a : A : \mathcal{U}_0$ can be | **Type-level** objects<br>$A : \kappa : \mathcal{U}_1$ can be | **Kind-level** objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be |
|---|---|---|---|
| 0-related<br>(het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$<br>$\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool} \to R \to R \to R} \mathit{if}_Y$ | $((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$<br>$([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$<br>... | $((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \mathrm{List}_4\ \mathcal{A}) \frown_0^{\mathrm{List}_\bullet\ \mathcal{A}} ([] : \mathrm{List}_6\ \mathcal{A})$<br>... |
| 1-related | n/a | $\big((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\big) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\mathrm{List}_\bullet\ A : \mathrm{List}_4\ A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\ A$<br>$R : (G : \mathrm{Grp}) \frown_1^{\mathrm{Grp}} (H : \mathrm{Grp})$<br>$R : (G : \mathrm{Grp}) \frown_1^{V} (M : \mathrm{Mon})$ | $\big((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\big) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\mathrm{List}_\bullet\ \kappa : \mathrm{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathrm{List}_6\ \kappa$<br>$\rho : (\alpha : \mathrm{Cat}) \frown_1^{\mathrm{Cat}} (\beta : \mathrm{Cat})$ |
| 2-related | n/a | n/a | $V : (\mathrm{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathrm{Mon} : \mathcal{U}_1)$ |

$$\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$$

| | **Value-level** objects<br>$a : A : \mathcal{U}_0$ can be | **Type-level** objects<br>$A : \kappa : \mathcal{U}_1$ can be | **Kind-level** objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be |
|---|---|---|---|
| 0-related<br>(het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$<br>$\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool}\to R\to R\to R} \mathit{if}_Y$ | $((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$<br>$([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$<br>... | $((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$<br>... |
| 1-related | n/a | $\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$<br>$R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$<br>$R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$ | $\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet\to\bullet\}}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$<br>$\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$ |
| 2-related | n/a | n/a | $V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$ |

$$\mathsf{List}_\bullet\ A : (\mathsf{List}_4\ A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathsf{List}_6\ A : \mathcal{U}_0)$$

| | **Value-level** objects<br>$a : A : \mathcal{U}_0$ can be | **Type-level** objects<br>$A : \kappa : \mathcal{U}_1$ can be | **Kind-level** objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be |
|---|---|---|---|
| 0-related<br>(het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$<br>$\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$ | $((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$<br>$([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$<br>... | $((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$<br>... |
| 1-related | n/a | $\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$<br>$R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$<br>$R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$ | $\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$<br>$\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$ |
| 2-related | n/a | n/a | $V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$ |

$$(G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$$
$$\cong$$
$$(\underline{R} : \underline{G} \frown_1^{\mathcal{U}_0} \underline{H}) \times (e_G \frown_0^{\underline{R}} e_H) \times (*_G \frown_0^{\underline{R} \to \underline{R} \to \underline{R}} *_H)$$

| | **Value-level** objects $a : A : \mathcal{U}_0$ can be | **Type-level** objects $A : \kappa : \mathcal{U}_1$ can be | **Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be |
|---|---|---|---|
| 0-related (het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ <br> $([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$ <br> $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ <br> $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$ | $((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ <br> $([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$ <br> $\ldots$ | $((\lambda\xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ <br> $([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$ <br> $\ldots$ |
| 1-related | n/a | $\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ <br> $\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ <br> $\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$ <br> $R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$ <br> $R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$ | $\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ <br> $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ <br> $\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$ <br> $\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$ |
| 2-related | n/a | n/a | $V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$ |

$$(G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$$
$$:=$$
$$(\underline{R} : \underline{G} \frown_1^{\mathcal{U}_0} \underline{M}) \times (e_G \frown_0^{\underline{R}} e_M) \times (*_G \frown_0^{\underline{R} \to \underline{R} \to \underline{R}} *_M)$$

| | **Value-level** objects<br>$a : A : \mathcal{U}_0$ can be | **Type-level** objects<br>$A : \kappa : \mathcal{U}_1$ can be | **Kind-level** objects<br>$\kappa : \mathcal{A} : \mathcal{U}_2$ can be |
|---|---|---|---|
| 0-related<br>(het. eq.) | $(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$<br>$([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$<br>$\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$<br>$\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$ | $((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$<br>$([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$<br>... | $((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$<br>$([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$<br>... |
| 1-related | n/a | $\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$<br>$\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$<br>$\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$<br>$R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$<br>$R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$ | $\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$<br>$\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$<br>$\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$<br>$\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$ |
| 2-related | n/a | n/a | $V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$ |

$$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$$