Degrees of Relatedness

A Unified Framework for Parametricity, Irrelevance, Ad Hoc Polymorphism, Intersections, Unions and Algebra in Dependent Type Theory

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Overview

- Parametricity
  - Intuition
  - In System F
  - In System Fω
  - In DTT
- Degrees of relatedness
  - Intro & known modalities
  - Structural modality
  - $\cap$ and $\cup$

Parametricity, intuitively

In System F, Fω, Haskell, . . . , **type parameters** are parametric.

- Only used for type-checking,
- Not inspected (e.g. no pattern matching),
- Same algorithm on all types.

Enforced by the type system.

Example

$g \quad : \quad \forall X.\text{Tree } X \to \text{List } X$

By parametricity:

$$\begin{tikzcd} A \arrow[d, "f"'] & \text{Tree } A \arrow[r, "g"] \arrow[d, "\text{Tree } f"'] & \text{List } A \arrow[d, "\text{List } f"] \\ B & \text{Tree } B \arrow[r, "g"] & \text{List } B \end{tikzcd}$$

irrespective of implementation.

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irrespective of implementation.

Intermezzo: Proof relevance

$R \in \mathrm{Rel}(A, B)$ if

- $R \subseteq A \times B$,
- $R : A \times B \to \mathrm{Prop}$,
- $R : A \times B \to \mathrm{Set}$,

$R(a, b)$ if

- $(a, b) \in R$,
- $* \in R(a, b)$,
- $r \in R(a, b)$.

Example (Isomorphic groups)

$(\cong) \in \mathrm{Rel}(\mathrm{Grp}, \mathrm{Grp})$
$\bar{x} \mapsto (\bar{x}, \bar{x}) : \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, so these groups are **isomorphic**.

Example (Related sets)

$\mathrm{Rel} \in \mathrm{Rel}(\mathrm{Set}, \mathrm{Set})$
$\mathit{ElemOf} \in \mathrm{Rel}(X, \mathrm{List}\, X)$, so these sets are **related**.
Always true, often in many ways!

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Parametricity in **System F**

(Reynolds, 1983)

System F: Type Judgements

$$X : *, \qquad Y : * \qquad \vdash \qquad X \times Y : *$$

Object semantics:

$$X \in \mathrm{Set}, \qquad Y \in \mathrm{Set} \qquad \Rightarrow \qquad X \times Y \in \mathrm{Set}$$

Relational semantics:

$$X_1 \in \mathrm{Set}, \qquad Y_1 \in \mathrm{Set} \qquad \Rightarrow \qquad X_1 \times Y_1 \in \mathrm{Set}$$

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Identity Extension Lemma (IEL)

...such that $\mathrm{Eq}_X \times \mathrm{Eq}_Y \cong \mathrm{Eq}_{X \times Y}$

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Propagating relations

Type formers propagate relations:

$\bar{X} \times \bar{Y}$ Componentwise,

$\bar{X} \to \bar{Y}$ For all x_1, x_2: $\bar{X}(x_1, x_2) \to \bar{Y}(f_1\, x_1, f_2\, x_2)$,

$\mathrm{List}\,\bar{X}$ Equal length, $\bar{X}$-related components,

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Either $t[X, p, q] = p$ or $t[X, p, q] = q$,
i.e. $\mathrm{Bool} \cong \forall X.X \to X \to X$

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When can we call a cross-type relation **“heterogeneous equality”**?

- Congruence (respected by everything),
- Extends equality: becomes equality in homogeneous case.

Term-relatedness à la Reynolds:

$\surd$ Is a congruence (prev. slide)

$\surd$ Identity extension

$\Rightarrow$ **Is a notion of het. equality.**

Type-relatedness à la Reynolds is **NOT**:
To prove relatedness is to give $\bar{X} \in \mathrm{Rel}(X_1, X_2)$
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Parametricity in **System Fω**

(Atkey, 2012)

Higher Kinds

IEL: Open types map preserve Eq.

If $X \in$ Set (sem. of $X : *$),
then $\text{Eq}_X \in \mathrm{Rel}(X, X)$

If $F \in \text{Set} \to \text{Set}$ (sem. of $F : * \to *$),
then what is $\text{Eq}_F \in (\mathrm{Rel} \to \mathrm{Rel})(F, F) = \mathrm{Rel}(X, Y) \to \mathrm{Rel}(FX, FY)$?

Nonsense: need to restructure model using **reflexivity.**

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IEL: Open types map preserve Eq.

If $X \in$ Set (sem. of $X : *$),
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Parametricity in **Dependent Type Theory**

DTT treats **types** and **terms** on equal footing, **BUT**

- Related **terms** are **het. equal**,
- Related **types** are **NOT**: $\mathrm{Rel} \neq \mathsf{Eq}$, $\bar{\kappa} \neq \mathsf{Eq}$.

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System $\mathbf{F}_\omega$:

Free Theorem (Yoneda lemma / Representation independence)

$$\forall (X : *).(X \to A) \to (X \to B) \qquad \cong \qquad A \to B$$

Dependent types:

$leak : (X : \mathcal{U}) \to (X \to A) \to (X \to \mathcal{U})$

$leak\, X\, f\, x = X$

Our solution (syntax-side):

$(\mathbf{par} \mid X : \mathcal{U}) \to (X \to A) \to (X \to \mathcal{U})$

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Let's have two relations

System F

Values can be related:

$$(s : S) \frown^R (t : T)$$

IEL: if $(s : A) \frown^A (t : A)$ then $s = t$
(heterogeneous equality)

Types can be related:

$$R : S \frown T$$

which gives meaning to

$$(s : S) \frown^R (t : T)$$

Dependent types

Things can be 0-related:

$$(s : S) \frown^R_0 (t : T)$$

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Will also write $s =^R t$ for $s \frown^R_0 t$.

Things can be 1-related:

$$(S : \kappa) \frown^\rho_1 (T : \lambda)$$

where $R : (S : \mathcal{U}) \frown^{\mathcal{U}}_1 (T : \mathcal{U})$ gives meaning to

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Continuity

$$\mathsf{List} : (\mathbf{con} \mid \mathcal{U}_0) \to \mathcal{U}_0$$

$$\begin{tikzcd}
X = Y \arrow[r] \arrow[d, "\mathsf{Eq}"'] \arrow[dr, phantom, "\mathsf{IEL}"] & \mathsf{List}\,X = \mathsf{List}\,Y \arrow[d, "\mathsf{Eq}"'] \\
X \frown_1 Y \arrow[r] & \mathsf{List}\,X \frown_1 \mathsf{List}\,Y
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In System F: $\to$

Parametricity

$$[] : (\mathbf{par} \mid X : \mathcal{U}_0) \to \mathsf{List}\,X$$

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This wasn't there in System F!

Continuity and Parametricity

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Degrees of Relatedness

- **Level -1 types:** $\top$ (propositions)
- **Level 0 types:** $= \to \top$
- **Level 1 types:** $= \to \frown_1 \to \top$
- **Level 2 types:** $= \to \frown_1 \to \frown_2 \to \top$
- ...

We can decouple **level** (predicativity) and **depth** (number of relations).

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- **Level 2 types:** $= \to \frown_1 \to \frown_2 \to \top$
- ...

We can decouple **level** (predicativity) and **depth** (number of relations).

Degrees of Relatedness

- **Level -1 types:** $\top$ (propositions)
- **Level 0 types:** $= \to \top$
- **Level 1 types:** $= \to \frown_1 \to \top$
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Degrees of Relatedness

- **Depth -1 types:** $\top$ (propositions)
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- **Depth 1 types:** $= \to \frown_1 \to \top$
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- $\ldots$

We can decouple **level** (predicativity) and **depth** (number of relations).

Continuity: $1 \to 1$

$$\mathrm{List} : (\mathbf{con} \mid \mathcal{U}_0) \to \mathcal{U}_0$$

$$\begin{tikzcd}
X = Y \arrow[r] \arrow[d, "\mathrm{Eq}"'] \arrow[dr, phantom, "\mathrm{IEL}"] & \mathrm{List}\,X = \mathrm{List}\,Y \arrow[d, "\mathrm{Eq}"'] \\
X \frown_1 Y \arrow[r] & \mathrm{List}\,X \frown_1 \mathrm{List}\,Y
\end{tikzcd}$$

Parametricity: $1 \to 0$

$$[] : (\mathbf{par} \mid X : \mathcal{U}_0) \to \mathrm{List}\,X$$

$$\begin{tikzcd}
X = Y \arrow[d, "\mathrm{Eq}"'] & \\
R : X \frown_1 Y \arrow[r] & []_X =^{\mathrm{List}\,R} []_Y
\end{tikzcd}$$

Definition

A **modality** $\mu : c \to d$ is any diagram from $c+1$ to $d+1$ relations that preserves het. equality.

Multi-mode type theory:

Licata & Shulman (2016), Licata, Shulman & Riley (2017)

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	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool}\to R\to R\to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$ $([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$ $\dots$
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathrm{List}_\bullet\ A : \mathrm{List}_4\ A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\ A$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be
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$$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$$

because

$$2+5 \equiv 7$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$ $([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$ $\ldots$
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathrm{List}_\bullet\ A : \mathrm{List}_4\ A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\ A$

$$((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$$

because

$$(\lambda X.X)\ \mathrm{Bool} \equiv \mathrm{Bool}$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$ $\dots$
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$$([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$$

where

$$\mathsf{List}_\bullet\ A \in \mathrm{Rel}(\mathsf{List}_4\ A, \mathsf{List}_6\ A)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be
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	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be
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$$(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$$
for some
$$R \in \mathrm{Rel}(\mathbb{N}, \mathrm{Bool})$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be
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$$(\mathit{if}_X : \mathrm{Bool} \to X \to X \to X) \frown_0^{\mathrm{Bool}\to R\to R\to R} (\mathit{if}_Y : \mathrm{Bool} \to Y \to Y \to Y)$$

for all

$$R \in \mathrm{Rel}(X, Y)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be
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$(a : A) \frown_i^{R} (b : B)$ is always w.r.t. $R : (A : \mathcal{U}_n) \frown_{i+1}^{\mathcal{U}_n} (B : \mathcal{U}_n)$

$$\Big((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\Big) := \mathrm{Rel}(A, B)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be
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$$\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$ $\ldots$
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$$\mathsf{List}_\bullet\ A : (\mathsf{List}_4\ A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathsf{List}_6\ A : \mathcal{U}_0)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$ $([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$ $\ldots$
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See paper for $\frown_2$ (as of **kind-level**) and higher.

Four laws:

- **Reflexivity:** $(a : A) \frown_i^A (a : A)$
- **Weakening:** $((a : A) \frown_i^R (b : B)) \to ((a : A) \frown_{i+1}^R (b : B))$
- **Dependency:** $(a : A) \frown_i^R (b : B)$ presumes $R : A \frown_{i+1}^{\mathcal{U}_n} B$
- **Identity extension:** $(a : A) \frown_0^A (b : A)$ means $a \equiv b : A$.

Ad hoc polymorphism

Law of excluded middle (wrong):

$$lem : (\textbf{par} \shortmid X : \mathcal{U}) \to X \uplus (X \to \text{Empty})$$

Free Theorem (contradiction!)

$$((\textbf{par} \shortmid X : \mathcal{U}) \to X) \uplus ((\textbf{par} \shortmid X : \mathcal{U}) \to X \to \text{Empty})$$

Ad hoc: $1 \to 0$

$$lem : (\textbf{hoc} \shortmid X : \mathcal{U}) \to X \uplus (X \to \text{Empty})$$

$$\begin{tikzcd} X = Y \arrow[r] \arrow[d] & lem\, X = lem\, Y \\ X \frown_1 Y & \end{tikzcd}$$

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$$\begin{tikzcd}
X = Y \arrow[r] \arrow[d] & lem\, X = lem\, Y \\
X \frown_1 Y &
\end{tikzcd}$$

Irrelevance

Irrelevance $:=$ ignored by definitional equality

Sized lists:

- $nil_X : (\mathbf{irr} \shortmid n : \mathbb{N}) \to (\mathbf{irr} \shortmid 0 < n) \to \mathrm{List}_n\, X$,
- $cons_X : (\mathbf{irr} \shortmid m\, n : \mathbb{N}) \to (\mathbf{irr} \shortmid m < n) \to X \to \mathrm{List}_m\, X \to \mathrm{List}_n\, X$

Two ways to annotate $[a]$:

- $as_2 :\equiv cons_A\, 2\, 5\, \_\, a\, (nil_A\, 2\, \_) : \mathrm{List}_5\, A$, $\quad nil_A\, 2\, \_ : \mathrm{List}_2\, A$
- $as_3 :\equiv cons_A\, 3\, 5\, \_\, a\, (nil_A\, 3\, \_) : \mathrm{List}_5\, A$, $\quad nil_A\, 3\, \_ : \mathrm{List}_3\, A$
- $cons_A \bullet \bullet\, \_\, a\, (nil_A \bullet \_) : \mathrm{List}_5\, A \Rightarrow as_2 \equiv as_3$,

Irrelevance is a **dependent** generalization of **constancy**.

Codomain $\mathrm{List}_{\sqcup}\, A$ must be **shape-irrelevant**.
Abel & Scherer (2012), example 2.8
Abel, Vezzosi & Winterhalter (2017)

Irrelevance $:=$ ignored by definitional equality

Sized lists:

- $\mathit{nil}_X : (\mathbf{irr} \shortmid n : \mathbb{N}) \to (\mathbf{irr} \shortmid 0 < n) \to \mathrm{List}_n\, X$,
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Two ways to annotate $[a]$:

- $as_2 :\equiv \mathit{cons}_A\, 2\, 5\, \_\, a\, (\mathit{nil}_A\, 2\, \_) : \mathrm{List}_5\, A$, $\quad \mathit{nil}_A\, 2\, \_ : \mathrm{List}_2\, A$
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Irrelevance is a **dependent** generalization of **constancy**.

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Abel & Scherer (2012), example 2.8

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Shape-Irrelevance and Irrelevance

Shape-irrelevance: $0 \to 1$

$$\mathsf{List}_{\sqcup}\, A : (\mathbf{shi} \shortmid \mathbb{N}) \to \mathcal{U}_0$$

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Composition of Modalities

Given

- $f : (\mu \mid A) \to B$,
- $g : (\nu \mid B) \to C$,

what is the modality of $g \circ f : (\nu \circ \mu \mid A) \to C$?

Example

$if_{(\mathrm{List}_4\, A)} : \mathrm{Bool} \to \mathrm{List}_4\, A \to \mathrm{List}_4\, A \to \mathrm{List}_4\, A$
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- We can ignore irrelevant parts.
- *if* uses first arg. parametrically.
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shi **irr** **par**

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All modalities at lowest levels

$(\mu \mid A) \to B$	$B : \mathcal{U}_0$ values	$B : \mathcal{U}_1$ types	$B : \mathcal{U}_n$
$A : \mathcal{U}_0$ values	**hoc**, **irr**	**hoc**, **shi**, **irr**	
$A : \mathcal{U}_1$ types	**hoc**, **par**, **irr**	**hoc**, **con**, **shi**, **par**, **shi&par**, **irr**	
$A : \mathcal{U}_m$			$\frac{(m+n+2)!}{(m+1)!(n+1)!}$

Algebra: The Structural Modality

Least fixpoint MuF of functor F : Type $\to$ Type:

- $\cong$ inductive type A with constructor $\alpha : F\,A \to A$
- $\cong$ initial F-algebra (i.e. type X with $\xi : F\,X \to X$)
- $\cong$ Church-encoding:

$$\forall \underbrace{X}_{\text{carrier}} .\ \underbrace{(FX \to X)}_{F\text{-algebra-structure}} \to X$$

View $\forall$ as **limit** operator.

Limit of everything is initial object.

Church encoding: MuF is limit of all F-algebras.

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Curry the Church-encoding?

Currying = collecting arguments in a (dependent) tuple.

System F

$$\forall X.(FX \to X) \to X$$
$$\cong^? (\exists X.FX \to X) \to X$$

Scope error: X is out of scope!

DTT

$$(\mathbf{par} \mid X : \mathcal{U}) \to (FX \to X) \to X$$
$$\cong^? \Big(\hat{X} : (\mathbf{par} \mid X : \mathcal{U}) \times (FX \to X)\Big) \to (\mathrm{fst}\, \hat{X})$$

Unsound: fst disrespects equality.

$$\top : \mathrm{Bool} \frown^{\mathcal{U}}_{1} \mathbb{N}$$
$$\bullet : \mathrm{true} =^{\top} 4$$
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Structurality to the rescue

$(\mathbf{par} \shortmid X : \mathcal{U}) \times (FX \to X)$ is a **data**type:

- **par** takes type X to value level,
- related types are identified,

F-algebras are **type**-level objects:

- X should stay at the type-level,
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$\mathrm{Alg}F = (X : \mathcal{U}) \times (\mathbf{str} \shortmid FX \to X)$

$$\begin{tikzcd}
R : A =^{\mathcal{U}} B \arrow[rr, bend left] & & (A, \alpha) =^{\mathrm{Alg}F} (B, \beta) \arrow[d] \\
R : A \frown_1^{\mathcal{U}} B \arrow[rr, bend left] & \alpha =^{FR \to R} \beta \arrow[ur] & (A, \alpha) \frown_1^{\mathrm{Alg}F} (B, \beta)
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Structurality

str : $d \to d + 1$

Structurality: how algebras depend on their structure.

Example (F-algebras)

$$\mathrm{Alg} F = (X : \mathcal{U}) \times (\mathbf{str} \mid FX \to X)$$

$$(\mathbf{par} \mid X : \mathcal{U}) \to (FX \to X) \to X \qquad \cong \qquad (\mathbf{par} \mid \hat{X} : \mathrm{Alg} F) \to \mathrm{fst}\,\hat{X}$$

$$\mathbf{par} \circ \mathbf{con} = \mathbf{par}$$

$$\mathbf{par} \circ \mathbf{str} = \mathbf{con}$$

Unions and Intersections

$$S \times T \cong (b : \mathrm{Bool}) \to \mathit{if}(b, S, T) \qquad \text{requires } R \Leftarrow S \frown^{\mathcal{U}}_{1} T$$
$$(s, t) \leftrightarrow \lambda b.\mathit{if}(b, s, t) \qquad \text{requires } \_ \Leftarrow s =^{R} t$$
$$\mathrm{fst} \leftrightarrow \lambda f.f\,\mathrm{true}$$
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$$S \uplus T \cong (b : \mathrm{Bool}) \times \mathit{if}(b, S, T) \qquad \text{requires } R \Leftarrow S \frown^{\mathcal{U}}_{1} T$$
$$\mathrm{inl}\, s \leftrightarrow (\mathrm{true}, s)$$
$$\mathrm{inr}\, t \leftrightarrow (\mathrm{false}, t)$$
$$\mathrm{case}(q, f, g) \leftrightarrow \mathit{if}(\mathrm{fst}\, q, f, g)\,(\mathrm{snd}\, q) \qquad \text{requires } \_ \Leftarrow f =^{R \to C} g$$

Definitional relatedness:

- Syntax: Type-checker can generate evidence.
- Semantics using erasure: Either hacky or WIP.

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$$S \cap T \cong (\mathbf{irr} \mid b : \mathrm{Bool}) \to \mathit{if}_{\mathbf{shi}}(b, S, T) \qquad \text{requires } R \Leftarrow S \frown^{\mathcal{U}}_{1} T$$

$$(s, t) \leftrightarrow \lambda b.\mathit{if}_{\mathbf{irr}}(b, s, t) \qquad \text{requires } \_ \Leftarrow s =^{R} t$$

$$\mathrm{fst} \leftrightarrow \lambda f.f\,\mathrm{true}$$

$$\mathrm{snd} \leftrightarrow \lambda f.f\,\mathrm{false}$$

$$S \uplus T \cong (b : \mathrm{Bool}) \times \mathit{if}(b, S, T) \qquad \text{requires } R \Leftarrow S \frown^{\mathcal{U}}_{1} T$$

$$\mathrm{inl}\, s \leftrightarrow (\mathrm{true}, s)$$

$$\mathrm{inr}\, t \leftrightarrow (\mathrm{false}, t)$$

$$\mathrm{case}(q, f, g) \leftrightarrow \mathit{if}(\mathrm{fst}\, q, f, g)\,(\mathrm{snd}\, q) \qquad \text{requires } \_ \Leftarrow f =^{R \to C} g$$

Definitional relatedness:

- Syntax: Type-checker can generate evidence.
- Semantics using erasure: Either hacky or WIP.

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The Cubical Model

The Model: Overview

Modelling multimode DTT:

- For every **mode** (depth) d, pick a **model** $\mathcal{D}$ of MLTT.
- For every **modality** $\mu : c \to d$, pick a **model morphism** $\mu : \mathcal{C} \to \mathcal{D}$.

Presheaf models:

- **Every presheaf cat.** models MLTT with $\Pi, \Sigma, \mathcal{U}, =$ with UIP, ...

Degrees of relatedness:

- Model d using **refl. graphs** with edges labelled $0, 1, \ldots, d$,
- Internal parametricity $\Rightarrow$ iterated graphs (= **cubical sets**),
- Restrict to **discrete** types: those that satisfy IEL,
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Fixing type formers

Fixing modal Σ-type:

- **Problem:** by default $(\mathrm{Bool}, \mathrm{true}) \neq (\mathbb{N}, 4) : (\mathbf{par} \shortmid X : \mathcal{U}) \times X$.
- **Solution:** take a quotient. Prove that eliminator still works. (fst becomes unsound.)

Fixing the universe:

- **Problem:** default Hofmann-Streicher universe $\mathcal{U}^{HS}$ is not discrete.
- **Default:** To prove $R : S \frown_i^{\mathcal{U}^{HS}} T$ is to define $s \frown_i^R t$.
  Wanted: To prove $R : S \frown_{i+1}^{\mathcal{U}} T$ is to define $s \frown_i^R t$.
- **Solution:** Define
  $S \frown_0^{\mathcal{U}} T$ as $S \equiv T$,
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Conclusion

- Unified framework (type system + presheaf model) for:
  - **parametricity**
  - **continuity**
  - **. irrelevance**
  - **.. shape-irrelevance**
  - **ad hoc polym.**
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- Understanding of
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  - **different nature of types** (depth)
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Implementations

Try it out!

`parametric` branch of Agda (by Andrea Vezzosi)

- Implements ParamDTT (depth 1 fragment)
- With Glue/Weld, you can prove free theorems internally
- `github.com/agda/agda/tree/parametric`
- `github.com/Saizan/parametric-demo`
- All 6 modalities $\mu : 1 \to 1$ are available, unlike in Nuyts, Vezzosi & Devriese (2017)

Degrees of Relatedness: In progress.

Future Work

Asymmetric relations:

- Proof-relevant subtyping,
- Directed type theory (synthetic category theory),
- Directed univalence : $(A \curvearrowright_1 B) \simeq (A \to B)$.

Erasure-based presheaf models for def. relatedness, $\cap$ and $\cup$.

Take home message

Describe function behaviour as action on degree of relatedness.
str, **con**, **par**, **hoc**, **shi**, **irr** are instances of this.

Thanks!

Questions?

Comparison with HoTT

Degrees of Relatedness	HoTT
functions **act** on $\frown_i$	functions **preserve** $\simeq$
equality as $\frown_0$	equality as $\simeq$
relational HITs[1]	groupoidal HITs
depth: $\mathcal{U}^d_\ell : \mathcal{U}^{d+1}_{\ell+1}$	h-level: $\mathcal{U}^h_\ell : \mathcal{U}^{h+1}_{\ell+1}$

[1] future work

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0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\, A) \frown_0^{\mathsf{List}_\bullet\, A} ([] : \mathsf{List}_6\, A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\,\mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\, \kappa) \frown_0^{\mathsf{List}_\bullet\, \kappa} ([] : \mathsf{List}_6\, \kappa)$ $\ldots$	$((\lambda \xi.\xi)\,\kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathsf{List}_4\, \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\, \mathcal{A}} ([] : \mathsf{List}_6\, \mathcal{A})$ $\ldots$
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because

$$2+5 \equiv 7$$

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where

$$\mathsf{List}_\bullet\, A \in \mathrm{Rel}(\mathsf{List}_4\, A, \mathsf{List}_6\, A)$$

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$$R \in \mathrm{Rel}(\mathbb{N}, \mathrm{Bool})$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathrm{List}_4\,A) \frown_0^{\mathrm{List}_\bullet\,A} ([] : \mathrm{List}_6\,A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool}\to R\to R\to R} \mathit{if}_Y$	$((\lambda X.X)\,\mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$ $([] : \mathrm{List}_4\,\kappa) \frown_0^{\mathrm{List}_\bullet\,\kappa} ([] : \mathrm{List}_6\,\kappa)$ …	$((\lambda\xi.\xi)\,\kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathrm{List}_4\,\mathcal{A}) \frown_0^{\mathrm{List}_\bullet\,\mathcal{A}} ([] : \mathrm{List}_6\,\mathcal{A})$ …
1-related	n/a	$\big((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\big) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathrm{List}_\bullet\,A : \mathrm{List}_4\,A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\,A$ $R : (G : \mathrm{Grp}) \frown_1^{\mathrm{Grp}} (H : \mathrm{Grp})$ $R : (G : \mathrm{Grp}) \frown_1^{V} (M : \mathrm{Mon})$	$\big((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\big) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet\to\bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathrm{List}_\bullet\,\kappa : \mathrm{List}_4\,\kappa \frown_1^{\mathcal{U}_1} \mathrm{List}_6\,\kappa$ $\rho : (\alpha : \mathrm{Cat}) \frown_1^{\mathrm{Cat}} (\beta : \mathrm{Cat})$
2-related	n/a	n/a	$V : (\mathrm{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathrm{Mon} : \mathcal{U}_1)$

$$(\mathit{if}_X : \mathrm{Bool} \to X \to X \to X) \frown_0^{\mathrm{Bool}\to R\to R\to R} (\mathit{if}_Y : \mathrm{Bool} \to Y \to Y \to Y)$$

for all

$$R \in \mathrm{Rel}(X, Y)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathrm{List}_4\, A) \frown_0^{\mathrm{List}_\bullet\, A} ([] : \mathrm{List}_6\, A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\,\mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$ $([] : \mathrm{List}_4\, \kappa) \frown_0^{\mathrm{List}_\bullet\, \kappa} ([] : \mathrm{List}_6\, \kappa)$ $\ldots$	$((\lambda \xi.\xi)\,\kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathrm{List}_4\, \mathcal{A}) \frown_0^{\mathrm{List}_\bullet\, \mathcal{A}} ([] : \mathrm{List}_6\, \mathcal{A})$ $\ldots$
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathrm{List}_\bullet\, A : \mathrm{List}_4\, A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\, A$ $R : (G : \mathrm{Grp}) \frown_1^{\mathrm{Grp}} (H : \mathrm{Grp})$ $R : (G : \mathrm{Grp}) \frown_1^{V} (M : \mathrm{Mon})$	$\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathrm{List}_\bullet\, \kappa : \mathrm{List}_4\, \kappa \frown_1^{\mathcal{U}_1} \mathrm{List}_6\, \kappa$ $\rho : (\alpha : \mathrm{Cat}) \frown_1^{\mathrm{Cat}} (\beta : \mathrm{Cat})$
2-related	n/a	n/a	$V : (\mathrm{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathrm{Mon} : \mathcal{U}_1)$

$$((\lambda X.X)\,\mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$$

because

$$(\lambda X.X)\,\mathrm{Bool} \equiv \mathrm{Bool}$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathrm{List}_4\, A) \frown_0^{\mathrm{List}_\bullet\, A} ([] : \mathrm{List}_6\, A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\, \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$ $([] : \mathrm{List}_4\, \kappa) \frown_0^{\mathrm{List}_\bullet\, \kappa} ([] : \mathrm{List}_6\, \kappa)$ ...	$((\lambda \xi.\xi)\, \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathrm{List}_4\, \mathcal{A}) \frown_0^{\mathrm{List}_\bullet\, \mathcal{A}} ([] : \mathrm{List}_6\, \mathcal{A})$ ...
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathrm{List}_\bullet\, A : \mathrm{List}_4\, A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\, A$ $R : (G : \mathrm{Grp}) \frown_1^{\mathrm{Grp}} (H : \mathrm{Grp})$ $R : (G : \mathrm{Grp}) \frown_1^{V} (M : \mathrm{Mon})$	$\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathrm{List}_\bullet\, \kappa : \mathrm{List}_4\, \kappa \frown_1^{\mathcal{U}_1} \mathrm{List}_6\, \kappa$ $\rho : (\alpha : \mathrm{Cat}) \frown_1^{\mathrm{Cat}} (\beta : \mathrm{Cat})$
2-related	n/a	n/a	$V : (\mathrm{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathrm{Mon} : \mathcal{U}_1)$

$$([] : \mathrm{List}_4\, \kappa) \frown_0^{\mathrm{List}_\bullet\, \kappa} ([] : \mathrm{List}_6\, \kappa)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathrm{List}_4\, A) \frown_0^{\mathrm{List}_\bullet\, A} ([] : \mathrm{List}_6\, A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$ $\forall R.if_X \frown_0^{\mathrm{Bool} \to R \to R \to R} if_Y$	$((\lambda X.X)\,\mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$ $([] : \mathrm{List}_4\, \kappa) \frown_0^{\mathrm{List}_\bullet\, \kappa} ([] : \mathrm{List}_6\, \kappa)$ $\dots$	$((\lambda \xi.\xi)\,\kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathrm{List}_4\, \mathcal{A}) \frown_0^{\mathrm{List}_\bullet\, \mathcal{A}} ([] : \mathrm{List}_6\, \mathcal{A})$ $\dots$
1-related	n/a	$\Big((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\Big) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathrm{List}_\bullet\, A : \mathrm{List}_4\, A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\, A$ $R : (G : \mathrm{Grp}) \frown_1^{\mathrm{Grp}} (H : \mathrm{Grp})$ $R : (G : \mathrm{Grp}) \frown_1^{V} (M : \mathrm{Mon})$	$\Big((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\Big) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathrm{List}_\bullet\, \kappa : \mathrm{List}_4\, \kappa \frown_1^{\mathcal{U}_1} \mathrm{List}_6\, \kappa$ $\rho : (\alpha : \mathrm{Cat}) \frown_1^{\mathrm{Cat}} (\beta : \mathrm{Cat})$
2-related	n/a	n/a	$V : (\mathrm{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathrm{Mon} : \mathcal{U}_1)$

$$\Big((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\Big) := \mathrm{Rel}(A, B)$$

$(a : A) \frown_i^{R} (b : B)$ is always w.r.t. $R : (A : \mathcal{U}_n) \frown_{i+1}^{\mathcal{U}_n} (B : \mathcal{U}_n)$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathrm{List}_4\, A) \frown_0^{\mathrm{List}_\bullet\, A} ([] : \mathrm{List}_6\, A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$ $\forall R.\, if_X \frown_0^{\mathrm{Bool}\to R\to R\to R} if_Y$	$((\lambda X.X)\,\mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$ $([] : \mathrm{List}_4\, \kappa) \frown_0^{\mathrm{List}_\bullet\, \kappa} ([] : \mathrm{List}_6\, \kappa)$ $\ldots$	$((\lambda \xi.\xi)\,\kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathrm{List}_4\, \mathcal{A}) \frown_0^{\mathrm{List}_\bullet\, \mathcal{A}} ([] : \mathrm{List}_6\, \mathcal{A})$ $\ldots$
1-related	n/a	$\big((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\big) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathrm{List}_\bullet\, A : \mathrm{List}_4\, A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\, A$ $R : (G : \mathrm{Grp}) \frown_1^{\mathrm{Grp}} (H : \mathrm{Grp})$ $R : (G : \mathrm{Grp}) \frown_1^{V} (M : \mathrm{Mon})$	$\big((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\big) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet\to\bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathrm{List}_\bullet\, \kappa : \mathrm{List}_4\, \kappa \frown_1^{\mathcal{U}_1} \mathrm{List}_6\, \kappa$ $\rho : (\alpha : \mathrm{Cat}) \frown_1^{\mathrm{Cat}} (\beta : \mathrm{Cat})$
2-related	n/a	n/a	$V : (\mathrm{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathrm{Mon} : \mathcal{U}_1)$

$$\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5:\mathbb{N}) \frown_0^{\mathbb{N}} (7:\mathbb{N})$ $([]:\mathrm{List}_4\,A) \frown_0^{\mathrm{List}_\bullet\,A} ([]:\mathrm{List}_6\,A)$ $\exists R.(5:\mathbb{N}) \frown_0^{R} (\mathrm{true}:\mathrm{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool}\to R\to R\to R} \mathit{if}_Y$	$((\lambda X.X)\,\mathrm{Bool}:\mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool}:\mathcal{U}_0)$ $([]:\mathrm{List}_4\,\kappa) \frown_0^{\mathrm{List}_\bullet\,\kappa} ([]:\mathrm{List}_6\,\kappa)$ $\ldots$	$((\lambda\xi.\xi)\,\kappa:\mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa:\mathcal{U}_1)$ $([]:\mathrm{List}_4\,\mathcal{A}) \frown_0^{\mathrm{List}_\bullet\,\mathcal{A}} ([]:\mathrm{List}_6\,\mathcal{A})$ $\ldots$
1-related	n/a	$\big((A:\mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B:\mathcal{U}_0)\big) := \mathrm{Rel}(A,B)$ $\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N}:\mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N}:\mathcal{U}_0)$ $\mathrm{List}_\bullet\,A : \mathrm{List}_4\,A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\,A$ $R:(G:\mathrm{Grp}) \frown_1^{\mathrm{Grp}} (H:\mathrm{Grp})$ $R:(G:\mathrm{Grp}) \frown_1^{V} (M:\mathrm{Mon})$	$\big((\kappa:\mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda:\mathcal{U}_1)\big) := \mathrm{Rel}(\kappa,\lambda)^{\{\bullet\to\bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0:\mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0:\mathcal{U}_1)$ $\mathrm{List}_\bullet\,\kappa : \mathrm{List}_4\,\kappa \frown_1^{\mathcal{U}_1} \mathrm{List}_6\,\kappa$ $\rho:(\alpha:\mathrm{Cat}) \frown_1^{\mathrm{Cat}} (\beta:\mathrm{Cat})$
2-related	n/a	n/a	$V:(\mathrm{Grp}:\mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathrm{Mon}:\mathcal{U}_1)$

$$\mathrm{List}_\bullet\,A : (\mathrm{List}_4\,A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathrm{List}_6\,A : \mathcal{U}_0)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\, A) \frown_0^{\mathsf{List}_\bullet\, A} ([] : \mathsf{List}_6\, A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\,\mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\, \kappa) \frown_0^{\mathsf{List}_\bullet\, \kappa} ([] : \mathsf{List}_6\, \kappa)$ ...	$((\lambda \xi.\xi)\,\kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathsf{List}_4\, \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\, \mathcal{A}} ([] : \mathsf{List}_6\, \mathcal{A})$ ...
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathsf{List}_\bullet\, A : \mathsf{List}_4\, A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\, A$ $R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$ $R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$	$\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathsf{List}_\bullet\, \kappa : \mathsf{List}_4\, \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\, \kappa$ $\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$
2-related	n/a	n/a	$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$

$$(G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$$
$$\cong$$
$$(\underline{R} : \underline{G} \frown_1^{\mathcal{U}_0} \underline{H}) \times (e_G \frown_0^{\underline{R}} e_H) \times (*_G \frown_0^{\underline{R} \to \underline{R} \to \underline{R}} *_H)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\, A) \frown_0^{\mathsf{List}_\bullet\, A} ([] : \mathsf{List}_6\, A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\,\mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\, \kappa) \frown_0^{\mathsf{List}_\bullet\, \kappa} ([] : \mathsf{List}_6\, \kappa)$ $\dots$	$((\lambda \xi.\xi)\,\kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathsf{List}_4\, \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\, \mathcal{A}} ([] : \mathsf{List}_6\, \mathcal{A})$ $\dots$
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathsf{List}_\bullet\, A : \mathsf{List}_4\, A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\, A$ $R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$ $R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$	$\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathsf{List}_\bullet\, \kappa : \mathsf{List}_4\, \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\, \kappa$ $\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$
2-related	n/a	n/a	$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$

$$(G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$$
$$:=$$
$$(\underline{R} : \underline{G} \frown_1^{\mathcal{U}_0} \underline{M}) \times (e_G \frown_0^{\underline{R}} e_M) \times (*_G \frown_0^{\underline{R} \to \underline{R} \to \underline{R}} *_M)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\, A) \frown_0^{\mathsf{List}_\bullet\, A} ([] : \mathsf{List}_6\, A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\,\mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\, \kappa) \frown_0^{\mathsf{List}_\bullet\, \kappa} ([] : \mathsf{List}_6\, \kappa)$ $\ldots$	$((\lambda \xi.\xi)\, \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathsf{List}_4\, \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\, \mathcal{A}} ([] : \mathsf{List}_6\, \mathcal{A})$ $\ldots$
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathsf{List}_\bullet\, A : \mathsf{List}_4\, A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\, A$ $R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$ $R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$	$\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathsf{List}_\bullet\, \kappa : \mathsf{List}_4\, \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\, \kappa$ $\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$
2-related	n/a	n/a	$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$

$$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$$