Degrees of Relatedness

A Unified Framework for Parametricity, Irrelevance, Ad Hoc Polymorphism, Intersections, Unions and Algebra in Dependent Type Theory

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Overview

- Parametricity
  - In System F
  - In System Fω
  - In dependent type theory
- Degrees of relatedness

Parametricity, intuitively

In System F, Fω, Haskell, . . . , **type parameters** are parametric.

- Only used for type-checking,
- Not inspected (e.g. no pattern matching),
- Same algorithm on all types.

Enforced by the type system.

Example

$g \quad : \quad \forall X.\text{Tree } X \to \text{List } X$

By parametricity:

irrespective of implementation.

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By parametricity:

$$
\begin{array}{ccccc}
A & \qquad & \text{Tree } A & \xrightarrow{g} & \text{List } A \\
\Big\downarrow f & & \text{Tree } f \Big\downarrow & & \Big\downarrow \text{List } f \\
B & & \text{Tree } B & \xrightarrow{g} & \text{List } B
\end{array}
$$

irrespective of implementation.

Parametricity in **System F**

(Reynolds, 1983)

System F: Type Judgements

$$X : *, \quad Y : * \quad \vdash \quad X \times Y : *$$

Object semantics:

$$X \in \mathrm{Set}, \quad Y \in \mathrm{Set} \quad \Rightarrow \quad X \times Y \in \mathrm{Set}$$

Relational semantics:

$$X_1 \in \mathrm{Set}, \quad Y_1 \in \mathrm{Set} \quad \Rightarrow \quad X_1 \times Y_1 \in \mathrm{Set}$$

$$X_2 \in \mathrm{Set}, \quad Y_2 \in \mathrm{Set} \quad \Rightarrow \quad X_2 \times Y_2 \in \mathrm{Set}$$

Identity Extension Lemma (IEL)

…such that $\mathrm{Eq}_X \times \mathrm{Eq}_Y \cong \mathrm{Eq}_{X \times Y}$

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$$\bar{X} \in \mathrm{Rel} \quad \bar{Y} \in \mathrm{Rel} \quad \Rightarrow \quad \bar{X} \times \bar{Y} \in \mathrm{Rel}$$

$$X_2 \in \mathrm{Set}, \quad Y_2 \in \mathrm{Set} \quad \Rightarrow \quad X_2 \times Y_2 \in \mathrm{Set}$$

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Propagating relations

Type formers propagate relations:

- $\bar{X} \times \bar{Y}$ Componentwise,
- $\bar{X} \to \bar{Y}$ For all x_1, x_2: $\bar{X}(x_1, x_2) \to \bar{Y}(f_1\ x_1, f_2\ x_2)$,
- List $\bar{X}$ Equal length, $\bar{X}$-related components,
- $\bar{X} \uplus \bar{Y}$ Same side, $\bar{X}$- or $\bar{Y}$-related.

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$$X : * \quad | \quad p : X, \quad q : X \quad \vdash \quad t[X, p, q] : X$$

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Free Theorem (Church Booleans)

Either $t[X, p, q] = p$ or $t[X, p, q] = q$,
i.e. $\mathrm{Bool} \cong \forall X.X \to X \to X$

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Intermezzo: Heterogeneous Equality

When can we call a cross-type relation **"heterogeneous equality"**?

- Congruence (respected by everything),
- Becomes equality in homogeneous case.

Term-relatedness à la Reynolds:

√ Is a congruence (prev. slide)

√ Identity extension

⇒ **Is a notion of het. equality.**

Type-relatedness à la Reynolds is **NOT**:
To prove relatedness is to give $\bar{X} \in \mathrm{Rel}(X_1, X_2)$
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Open types map Rel-related types to Rel-related types:

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Open terms map Rel-related types
and het. equal values to het. equal values:

$$\begin{array}{ccccc} X_1 \in \mathrm{Set}, & p_1 \in X_1, & q_1 \in X_1 & \Rightarrow & t[X_1, p_1, q_1] \in X_1 \\ \big|\,\bar{X}\in\mathrm{Rel} & \|\,\bar{p}\in\bar{X} & \|\,\bar{q}\in\bar{X} & \Rightarrow & \|\,t[\bar{X},\bar{p},\bar{q}]\in\bar{X} \\ X_2 \in \mathrm{Set}, & p_2 \in X_2, & q_2 \in X_2 & \Rightarrow & t[X_2, p_2, q_2] \in X_2 \end{array}$$

Parametricity Summarized

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Parametricity in **System Fω**

(Atkey, 2012)

Higher Kinds

Kind κ	Obj. semantics κ	Rel. semantics $\frown\kappa$
$*$	Set	Rel
$* \times *$	Set $\times$ Set	Rel $\times$ Rel
$* \to *$	Set $\to$ Set $\mathrm{Eq}_{\mathrm{Set}} \longrightarrow \mathrm{Eq}_{\mathrm{Set}}$ Eq $\downarrow$ IEL $\downarrow$ Eq $\mathrm{Rel} \longrightarrow \mathrm{Rel}$	Rel $\to$ Rel
$\kappa \to \lambda$	Set $\to$ Set $\mathrm{Eq}_\kappa \longrightarrow \mathrm{Eq}_\lambda$ $\downarrow$ $\downarrow$ $\frown\kappa \longrightarrow \frown\lambda$	$\frown\kappa \to \frown\lambda$

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Parametricity in **Dependent Type Theory**

DTT treats **types** and **terms** on equal footing, **BUT**

- Related **terms** are **het. equal**,
- Related **types** are **NOT**: $\mathrm{Rel} \neq \mathrm{Eq}$.

Mainstream approach: Ignore this fact

Takeuti (2001), Krishnaswami & Dreyer (2013), Atkey, Ghani & Johann (2014)

$\Rightarrow$ Free theorems can break for large types.

Our approach:

- Equip types with **both** $=$ and $\frown$,
- Modality on $(\mu \mid x : A) \to B\ x$ says how function acts on relations,
- Free theorems hold for parametric functions.

Nuyts, Vezzosi & Devriese (2017)

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Mainstream approach: Ignore this fact

Takeuti (2001), Krishnaswami & Dreyer (2013), Atkey, Ghani & Johann (2014)

$\Rightarrow$ Free theorems can break for large types.

Our approach:

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Nuyts, Vezzosi & Devriese (2017)

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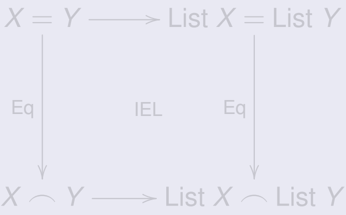
Continuity and Parametricity

Continuity

$$\text{List} : (\textbf{con} \mid \mathcal{U}_0) \to \mathcal{U}_0$$



In System F: $\to$

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This wasn't there in System F!

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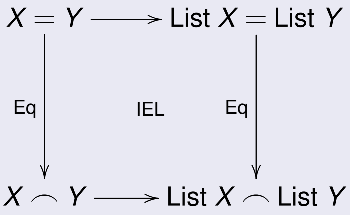

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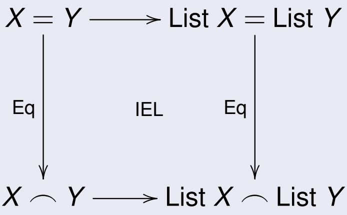

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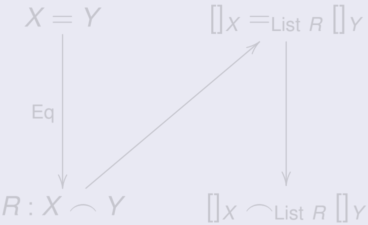

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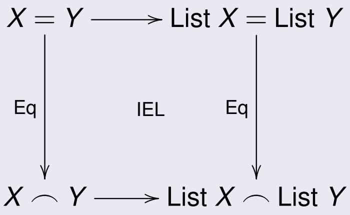

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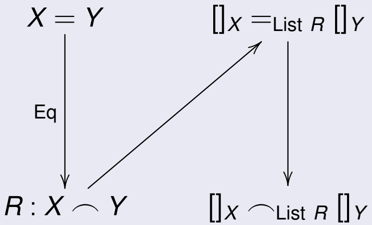

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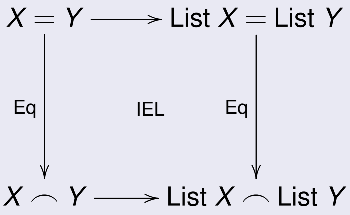

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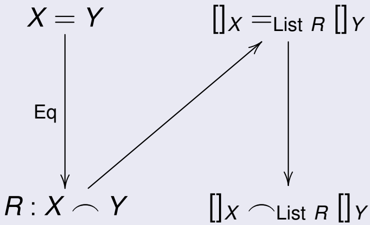

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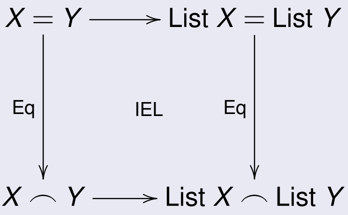

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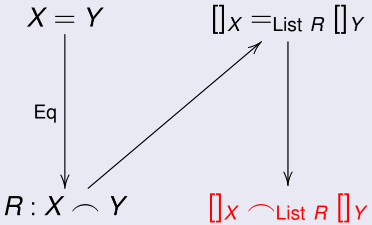

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Degrees of Relatedness

- **Level -1 types:** $\top$ (propositions)
- **Level 0 types:** $= \to \top$
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We can decouple **level** (predicativity) and **depth** (number of relations).

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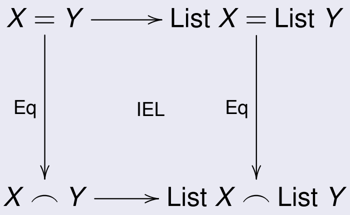

Parametricity: 1 → 0

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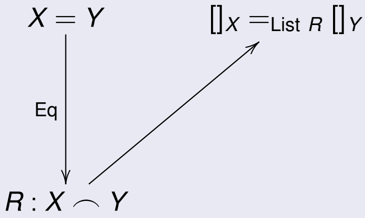

Ad hoc polymorphism

Law of excluded middle (wrong):

$$lem : (\mathbf{par} \shortmid X : \mathcal{U}) \to X \uplus (X \to \text{Empty})$$

Free Theorem (contradiction!)

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Ad hoc: $1 \to 0$

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Irrelevance

Irrelevance $:=$ ignored by definitional equality

Sized lists:

- $nil_X : (\textbf{irr} \mid n : \mathbb{N}) \to \mathrm{List}_n\ X$,
- $cons_X : (\textbf{irr} \mid m\ n : \mathbb{N}) \to (\textbf{irr} \mid m < n) \to X \to \mathrm{List}_m\ X \to \mathrm{List}_n\ X$

Irrelevance is a **dependent** generalization of **constancy**.

Codomain $\mathrm{List}_{\sqcup}\ A$ must be **shape-irrelevant**.

Abel & Scherer (2012), example 2.8

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Shape-irrelevance: 0 → 1

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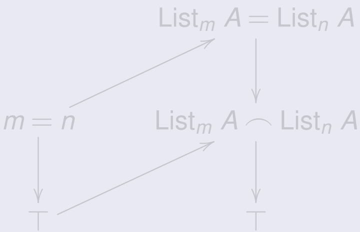

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Shape-Irrelevance and Irrelevance

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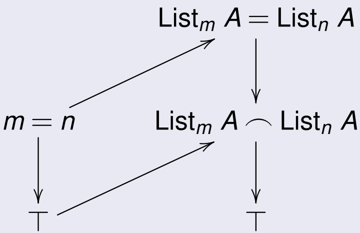

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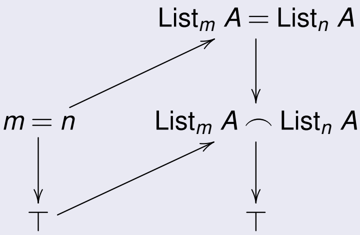

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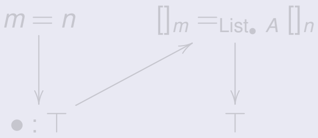

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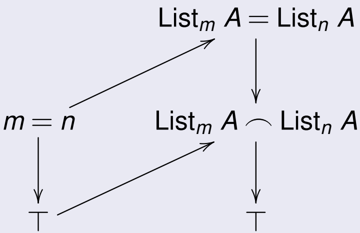

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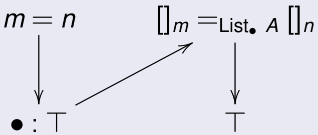

Take home message

Describe function behaviour as action on degree of relatedness.
con, **par**, **hoc**, **shi**, **irr** are instances of this.

Thanks!

Further questions?

Breaking free theorems in DTT

System F_ω:

Free Theorem

$$\forall X.(X \to A) \to (X \to B) \quad \cong \quad A \to B$$

Dependent types:

$leak : (X : \mathcal{U}) \to (X \to A) \to (X \to \mathcal{U})$

$leak\ X\ f\ x = X$

Our solution:

$(\textbf{par} \mid X : \mathcal{U}) \to (X \to A) \to (X \to \mathcal{U})$

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All modalities at lowest levels

$(\mu \mid A) \to B$	$B : \mathcal{U}_0$ values	$B : \mathcal{U}_1$ types	$B : \mathcal{U}_n$
$A : \mathcal{U}_0$ values	**hoc**, **irr**	**hoc**, **shi**, **irr**	
$A : \mathcal{U}_1$ types	**hoc**, **par**, **irr**	**hoc**, **con**, **shi**, **par**, **shi&par**, **irr**	
$A : \mathcal{U}_m$			$\frac{(m+n+2)!}{(m+1)!(n+1)!}$

Comparison with HoTT

Degrees of Relatedness	HoTT
functions **act** on $\frown_i$	functions **preserve** $\simeq$
equality as $\frown_0$	equality as $\simeq$
relational HITs[1]	groupoidal HITs
depth: $\mathcal{U}_\ell^d : \mathcal{U}_{\ell+1}^{d+1}$	h-level: $\mathcal{U}_\ell^h : \mathcal{U}_{\ell+1}^{h+1}$

[1] future work

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$ $([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$ ...	$((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathrm{List}_4\ \mathcal{A}) \frown_0^{\mathrm{List}_\bullet\ \mathcal{A}} ([] : \mathrm{List}_6\ \mathcal{A})$ ...
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathrm{List}_\bullet\ A : \mathrm{List}_4\ A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\ A$ $R : (G : \mathrm{Grp}) \frown_1^{\mathrm{Grp}} (H : \mathrm{Grp})$ $R : (G : \mathrm{Grp}) \frown_1^{V} (M : \mathrm{Mon})$	$\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathrm{List}_\bullet\ \kappa : \mathrm{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathrm{List}_6\ \kappa$ $\rho : (\alpha : \mathrm{Cat}) \frown_1^{\mathrm{Cat}} (\beta : \mathrm{Cat})$
2-related	n/a	n/a	$V : (\mathrm{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathrm{Mon} : \mathcal{U}_1)$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool}\to R\to R\to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$ …	$((\lambda\xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$ …
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$ $R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$ $R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$	$\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet\to\bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$ $\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$
2-related	n/a	n/a	$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$

$$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$$

because

$$2+5 \equiv 7$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$ $\ldots$	$((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$ $\ldots$
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$ $R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$ $R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$	$\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$ $\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$
2-related	n/a	n/a	$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$

$$([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$$

where

$$\mathsf{List}_\bullet\ A \in \mathrm{Rel}(\mathsf{List}_4\ A, \mathsf{List}_6\ A)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$ ...	$((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$ ...
1-related	n/a	$\big((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\big) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$ $R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$ $R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$	$\big((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\big) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$ $\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$
2-related	n/a	n/a	$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$

$$(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$$
for some
$$R \in \mathrm{Rel}(\mathbb{N}, \mathsf{Bool})$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool}\to R\to R\to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$ $([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$ ...	$((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathrm{List}_4\ \mathcal{A}) \frown_0^{\mathrm{List}_\bullet\ \mathcal{A}} ([] : \mathrm{List}_6\ \mathcal{A})$ ...
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathrm{List}_\bullet\ A : \mathrm{List}_4\ A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\ A$ $R : (G : \mathrm{Grp}) \frown_1^{\mathrm{Grp}} (H : \mathrm{Grp})$ $R : (G : \mathrm{Grp}) \frown_1^{V} (M : \mathrm{Mon})$	$\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet\to\bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathrm{List}_\bullet\ \kappa : \mathrm{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathrm{List}_6\ \kappa$ $\rho : (\alpha : \mathrm{Cat}) \frown_1^{\mathrm{Cat}} (\beta : \mathrm{Cat})$
2-related	n/a	n/a	$V : (\mathrm{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathrm{Mon} : \mathcal{U}_1)$

$$(\mathit{if}_X : \mathrm{Bool} \to X \to X \to X) \frown_0^{\mathrm{Bool}\to R\to R\to R} (\mathit{if}_Y : \mathrm{Bool} \to Y \to Y \to Y)$$

for all

$$R \in \mathrm{Rel}(X, Y)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$ …	$((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$ …
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$ $R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$ $R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$	$\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$ $\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$
2-related	n/a	n/a	$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$

$$((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$$

because

$$(\lambda X.X)\ \mathsf{Bool} \equiv \mathsf{Bool}$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$ ...	$((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$ ...
1-related	n/a	$\big((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\big) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$ $R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$ $R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$	$\big((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\big) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$ $\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$
2-related	n/a	n/a	$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$

$$([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$ ...	$((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$ ...
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$ $R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$ $R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$	$\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$ $\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$
2-related	n/a	n/a	$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$

$(a : A) \frown_i^{R} (b : B)$ is always w.r.t. $R : (A : \mathcal{U}_n) \frown_{i+1}^{\mathcal{U}_n} (B : \mathcal{U}_n)$

$$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool}\to R\to R\to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$ $\dots$	$((\lambda\xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$ $\dots$
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A,B)$ $\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$ $R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$ $R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$	$\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa,\lambda)^{\{\bullet\to\bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$ $\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$
2-related	n/a	n/a	$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$

$$\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathrm{List}_4\ A) \frown_0^{\mathrm{List}_\bullet\ A} ([] : \mathrm{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathrm{true} : \mathrm{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathrm{Bool}\to R\to R\to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathrm{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathrm{Bool} : \mathcal{U}_0)$ $([] : \mathrm{List}_4\ \kappa) \frown_0^{\mathrm{List}_\bullet\ \kappa} ([] : \mathrm{List}_6\ \kappa)$ ...	$((\lambda\xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathrm{List}_4\ \mathcal{A}) \frown_0^{\mathrm{List}_\bullet\ \mathcal{A}} ([] : \mathrm{List}_6\ \mathcal{A})$ ...
1-related	n/a	$\big((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\big) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathrm{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathrm{List}_\bullet\ A : \mathrm{List}_4\ A \frown_1^{\mathcal{U}_0} \mathrm{List}_6\ A$ $R : (G : \mathrm{Grp}) \frown_1^{\mathrm{Grp}} (H : \mathrm{Grp})$ $R : (G : \mathrm{Grp}) \frown_1^{V} (M : \mathrm{Mon})$	$\big((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\big) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet\to\bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathrm{List}_\bullet\ \kappa : \mathrm{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathrm{List}_6\ \kappa$ $\rho : (\alpha : \mathrm{Cat}) \frown_1^{\mathrm{Cat}} (\beta : \mathrm{Cat})$
2-related	n/a	n/a	$V : (\mathrm{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathrm{Mon} : \mathcal{U}_1)$

$$\mathrm{List}_\bullet\ A : (\mathrm{List}_4\ A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathrm{List}_6\ A : \mathcal{U}_0)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$ ...	$((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$ ...
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$ $R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$ $R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$	$\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$ $\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$
2-related	n/a	n/a	$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$

$$(G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$$
$$\cong$$
$$(\underline{R} : \underline{G} \frown_1^{\mathcal{U}_0} \underline{H}) \times (e_G \frown_0^{\underline{R}} e_H) \times (*_G \frown_0^{\underline{R} \to \underline{R} \to \underline{R}} *_H)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$ ...	$((\lambda \xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$ ...
1-related	n/a	$\big((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\big) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$ $R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$ $R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$	$\big((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\big) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$ $\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$
2-related	n/a	n/a	$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$

$$(G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$$
$$:=$$
$$(\underline{R} : \underline{G} \frown_1^{\mathcal{U}_0} \underline{M}) \times (e_G \frown_0^{\underline{R}} e_M) \times (*_G \frown_0^{\underline{R} \to \underline{R} \to \underline{R}} *_M)$$

	Value-level objects $a : A : \mathcal{U}_0$ can be	**Type-level** objects $A : \kappa : \mathcal{U}_1$ can be	**Kind-level** objects $\kappa : \mathcal{A} : \mathcal{U}_2$ can be
0-related (het. eq.)	$(2+5 : \mathbb{N}) \frown_0^{\mathbb{N}} (7 : \mathbb{N})$ $([] : \mathsf{List}_4\ A) \frown_0^{\mathsf{List}_\bullet\ A} ([] : \mathsf{List}_6\ A)$ $\exists R.(5 : \mathbb{N}) \frown_0^{R} (\mathsf{true} : \mathsf{Bool})$ $\forall R.\mathit{if}_X \frown_0^{\mathsf{Bool} \to R \to R \to R} \mathit{if}_Y$	$((\lambda X.X)\ \mathsf{Bool} : \mathcal{U}_0) \frown_0^{\mathcal{U}_0} (\mathsf{Bool} : \mathcal{U}_0)$ $([] : \mathsf{List}_4\ \kappa) \frown_0^{\mathsf{List}_\bullet\ \kappa} ([] : \mathsf{List}_6\ \kappa)$ ...	$((\lambda\xi.\xi)\ \kappa : \mathcal{U}_1) \frown_0^{\mathcal{U}_1} (\kappa : \mathcal{U}_1)$ $([] : \mathsf{List}_4\ \mathcal{A}) \frown_0^{\mathsf{List}_\bullet\ \mathcal{A}} ([] : \mathsf{List}_6\ \mathcal{A})$ ...
1-related	n/a	$\left((A : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (B : \mathcal{U}_0)\right) := \mathrm{Rel}(A, B)$ $\mathbb{N} := \mathsf{Eq}_{\mathbb{N}} : (\mathbb{N} : \mathcal{U}_0) \frown_1^{\mathcal{U}_0} (\mathbb{N} : \mathcal{U}_0)$ $\mathsf{List}_\bullet\ A : \mathsf{List}_4\ A \frown_1^{\mathcal{U}_0} \mathsf{List}_6\ A$ $R : (G : \mathsf{Grp}) \frown_1^{\mathsf{Grp}} (H : \mathsf{Grp})$ $R : (G : \mathsf{Grp}) \frown_1^{V} (M : \mathsf{Mon})$	$\left((\kappa : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\lambda : \mathcal{U}_1)\right) := \mathrm{Rel}(\kappa, \lambda)^{\{\bullet \to \bullet\}}$ $\mathcal{U}_0 : (\mathcal{U}_0 : \mathcal{U}_1) \frown_1^{\mathcal{U}_1} (\mathcal{U}_0 : \mathcal{U}_1)$ $\mathsf{List}_\bullet\ \kappa : \mathsf{List}_4\ \kappa \frown_1^{\mathcal{U}_1} \mathsf{List}_6\ \kappa$ $\rho : (\alpha : \mathsf{Cat}) \frown_1^{\mathsf{Cat}} (\beta : \mathsf{Cat})$
2-related	n/a	n/a	$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$

$$V : (\mathsf{Grp} : \mathcal{U}_1) \frown_2^{\mathcal{U}_1} (\mathsf{Mon} : \mathcal{U}_1)$$