Transpension: The Right Adjoint to the Pi-Type

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Presheaf models of dependent type theory [Hof97, HS97] have been successfully applied to model HoTT [BCH14, CMS20, CCHM17, Hub16, KLV12, Ort18, OP18], parametricity [AGJ14, BCM15, ND18a, NVD17], and directed, guarded [BM20] and nominal [Pit13, §6.3] type theory, as well as combinations of these [BBC+19, CH20, RS17, WL20].[1] There has been considerable interest in internalizing aspects of these presheaf models, either to make the resulting language more expressive, or in order to carry out further reasoning internally, allowing greater abstraction and sometimes automated verification. While the constructions of presheaf models largely follow a common pattern, approaches towards internalization do not. Throughout the literature, various internal presheaf operators ($\surd$ [LOPS18], Φ/extent and Ψ/Gel [BCM15, Mou16, CH20], Glue and Weld [CCHM17, NVD17], mill [ND18b], the strictness axiom [OP18] and locally fresh names [PMD15]) can be found with little or no analysis of their relative expressiveness. Moreover, some of these require that variables whose type is a *shape* (representable presheaf, e.g. an interval) be used affinely.

Three years ago [ND19], we proposed the addition of a *transpension* type $\between u : \mathrm{Ty}(\Gamma) \to \mathrm{Ty}(\Gamma, u : \mathbb{U})$, right adjoint to structural or substructural universal quantification over a shape $\forall(u : \mathbb{U}) : \mathrm{Ty}(\Gamma, u : \mathbb{U}) \to \mathrm{Ty}(\Gamma)$, as a means of internalizing the peculiarities of presheaf models in general. Each of the aforementioned internal operators can be implemented from transpension, strictness and/or a pushout type former.[2] The transpension type has a structure reminiscent of a dependent version of the suspension type in HoTT [Uni13, §6.5]. In topoi, a right adjoint to structural quantification $\Pi(u : \mathbb{U})$ has already been considered by Yetter [Yet87], who named it ∇ and proved that it is definable from the amazing right adjoint $\surd$.

The structural transpension coquantifier $\between u$ is part of a sequence of adjoints $\Sigma\, u \dashv \Omega\, u \dashv \Pi\, u \dashv \between u$, preceded by the Σ-type, weakening and the Π-type. Adjointness of the first three is provable from the structural rules of type theory, but it is not immediately clear how to add typing rules for a further adjoint. Birkedal et al. [BCM+20] explain how to add a single modality that has a left adjoint in the semantics. If we want to have two or more adjoint modalities internally, then we can use a multimodal type system such as MTT [GKNB21, GKNB20].

In a paper currently under review [ND21] we present an extensional type system, an extension of an instance of MTT, featuring the transpension type as a modality and backed by a denotational model. Each internal modality in MTT needs a semantic left adjoint, so we can only internalize $\Omega\, u \dashv \Pi\, u \dashv \between u$. A drawback which we accept (as a challenge for future work), is that $\Omega\, u$ and $\Pi\, u$ become modalities which are a bit more awkward to deal with than ordinary weakening and Π-types. Below, we explain the main ideas of our approach, without reiterating the benefits and applications of the transpension type [ND19].

Shapes and Multipliers Transpension is right adjoint to universal quantification over a shape. Because we want to support both structural (cartesian) and substructural (e.g. affine) quantification, our definition of shape needs to be a bit more general than just 'a representable object' of the presheaf model $\mathrm{Psh}(\mathcal{W})$ which would be essentially, via the Yoneda-embedding, an object of $\mathcal{W}$. Instead, a shape $\mathbb{U}$ will be modelled by an arbitrary endofunctor denoted $\sqcup \ltimes U : \mathcal{W} \to \mathcal{W}$,[3] dubbed a *multiplier*. We write U for a chosen object isomorphic to $\top \ltimes U$ so that we can always project $\pi_2 : W \ltimes U \to U$.

Examples of such shapes are: the interval $\mathbb{I}$ in affine and cartesian cubical models of HoTT and/or parametricity, the sort of names in nominal type theory [PMD15], the sort of clocks of a

[1]We omit models that are not explicitly structured as presheaf models [AHH18, LH11, Nor19].
[2]For locally fresh names, we only have a heuristic translation.
[3]In a technical report [Nuy21], we generalize beyond endofunctors.

given finite longevity in guarded type theory [BM20], and the twisted prism functor [PK20] which we believe is important for directed type theory.

Because arbitrary endofunctors are a bit too general to obtain many useful results, we introduce some criteria to classify multipliers. The multiplier gives rise to a functor $Ⅎ_U : \mathcal{W} \to \mathcal{W}/U : W \mapsto (W \ltimes U, \pi_2)$ to the slice category over U. We say that $\sqcup \ltimes U$ is

- *semicartesian* if it is copointed, i.e. there is a first projection $\pi_1 : W \ltimes U \to W$,
- *cartesian* if it is a cartesian product,
- *cancellative* if $Ⅎ_U$ is faithful (equivalently if $\sqcup \ltimes U$ is),
- *affine* if $Ⅎ_U$ is full (which rules out being cartesian unless $U \cong \top$),
- *connection-free* if $Ⅎ_U$ is essentially surjective on slices (V, φ) such that $\varphi : V \to U$ is *dimensionally split*, which in most cases just means split epi,
- *quantifiable* if $Ⅎ_U$ has a left adjoint $\exists_U : \mathcal{W}/U \to \mathcal{W}$ (i.e. if $\sqcup \ltimes U$ is a local right adjoint).

For the properties of the example multipliers, we refer to the paper [ND21].

Modes are Shape Contexts Every MTT judgement $p \mid \Gamma \vdash J$ is stated at some mode p, and modalities $\mu : p \to q$ have a domain and a codomain mode. The introduction rule for the modal type looks like this:

$$\frac{p \mid \Gamma, \text{🔒}_\mu \vdash a : A \qquad \mu : p \to q}{q \mid \Gamma \vdash \mathsf{mod}_\mu\, a : \langle \mu \mid A \rangle}$$

Typically (but not necessarily) every mode p will be modelled by a presheaf category $[\![p]\!]$ and every modality $\mu : p \to q$ will be modelled by a DRA [BCM+20] $[\![\text{🔒}_\mu]\!] \dashv [\![\mu]\!] : [\![p]\!] \to [\![q]\!]$.

A complication is that the modalities that we need, bind or depend on a variable, a phenomenon which is not supported by MTT. We solve this by grouping shape variables such as $u : \mathbb{U}$ in a *shape context* which is not considered part of the type-theoretic context but instead serves as the *mode* of the judgement. Formally, we define a shape context as any presheaf over $\mathcal{W}$, but in practice shape contexts will be denoted $(u_1 : \mathbb{U}_1, \dots, u_n : \mathbb{U}_n)$ and obtained by applying (the left Kan extensions of) the corresponding multipliers to the terminal presheaf. Judgements in shape context Ξ are then interpreted in presheaves over the category of elements $\mathcal{W}/\Xi$, i.e. dependent presheaves over Ξ.

Modalities As modalities $\mu : \Xi_1 \to \Xi_2$, we take *all* DRAs from $\mathrm{Psh}(\mathcal{W}/\Xi_1)$ to $\mathrm{Psh}(\mathcal{W}/\Xi_2)$. Again, a few specific ones are of special interest:

Modalities for substitution. A shape substitution (presheaf morphism) $\sigma : \Xi_1 \to \Xi_2$ leads to a functor $\Sigma^{/\sigma} : \mathcal{W}/\Xi_1 \to \mathcal{W}/\Xi_2$ which, by left Kan extension, precomposition and right Kan extension, leads to a triple of adjoint functors $\Sigma^{\sigma|} \dashv \Omega^{\sigma|} \dashv \Pi^{\sigma|} : \mathrm{Psh}(\mathcal{W}/\Xi_1) \to \mathrm{Psh}(\mathcal{W}/\Xi_2)$, the latter two of which can be internalized as modalities $\Omega\,\sigma \dashv \Pi\,\sigma$. Of course $\Omega\,\sigma$ is the substitution modality. In case σ is really a weakening $\pi : (\Xi, u : \mathbb{U}) \to \Xi$ over a semicartesian shape $\mathbb{U}$, then we write $\Omega\,u \dashv \Pi(u : \mathbb{U})$ and these stand for weakening and the Π-type.

Modalities for (co)quantification. A quantifiable multiplier gives rise to functors $\exists_U^{/\Xi} \dashv Ⅎ_U^{/\Xi} : \mathcal{W}/\Xi \to \mathcal{W}/(\Xi, u : \mathbb{U})$, whence by Kan extension and precomposition a quadruple of adjoint functors $\exists_U^{\Xi|} \dashv Ⅎ_U^{\Xi|} \dashv \forall_U^{\Xi|} \dashv \between_U^{\Xi|} : \mathrm{Psh}(\mathcal{W}/\Xi) \to \mathrm{Psh}(\mathcal{W}/(\Xi, u : \mathbb{U}))$, the latter three of which can be internalized as $Ⅎ\,u \dashv \forall(u : \mathbb{U}) \dashv \between\,u$ which stand for fresh weakening, substructural quantification and transpension.

Theorem If a multiplier $\sqcup \ltimes U$ is:

1. semicartesian, then we get morphisms $\mathsf{spoil}_u : Ⅎ\,u \Rightarrow \Omega\,u$ and (hence) $\mathsf{cospoil}_u : \Pi\,u \Rightarrow \forall\,u$,
2. cartesian, then spoil_u and (hence) $\mathsf{cospoil}_u$ are isomorphisms,
3. cancellative and affine, then $\forall\,u \circ Ⅎ\,u \cong 1$ and (hence) $\forall\,u \circ \between\,u \cong 1$,
4. cancellative, affine and connection-free, then the transpension type admits a pattern-matching eliminator not unlike that of the suspension type; equivalently, Φ/extent is then sound.

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