# Models of Naturality Pretype Theory: Technical Report v0.2

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# Contents

# Chapter 1

# Base Categories and Modes

In this chapter, we establish the object part of the mode theory for naturality pretype theory and its model. Modes will be **anpolarity masks** $\vec{a}$ (section 1.1) which will be modelled in presheaf categories over certain base categories. We propose two families of base categories, parametrized by the mask $\vec{a}$, and both are built using the category $\mathsf{JetSet}(\vec{a})$ of **jet sets** over $\vec{a}$ (section 1.2). Jet sets themselves are not suitable as a base category, e.g. because they are closed under coproducts, which the Yoneda-embedding will never preserve.

The more complex but better behaved family of base categories are categories of **jet cubes** $\mathsf{JetCube}^{⌑}_{M}(\mathsf{f}, \vec{a})$ (section 1.4). While we can get a grasp on jet cubes by characterizing their morphisms using a calculus (section 1.4.3), this calculus is relatively complex and its soundness and completeness proofs are even more complex.

For these reasons, we alternatively propose to use categories of **jet topes** $\mathsf{JetTope}(\vec{a})$ (section 1.3). These are full subcategories of $\mathsf{JetSet}(\vec{a})$ on objects that satisfy certain somewhat arbitrary well-behavedness criteria. The main purpose of the categories of jet topes is to be workable for most basic purposes without being as complex as the categories of jet cubes.

Upon a first lecture, readers may choose to omit section 1.4 on jet cubes.

Finally in section 3.1, we introduce a number of operations on anpolarity masks which serve to insert bridge and/or path relations for every degree. These will be both a technical device and a source of interesting modes in practice.

## 1.1 Anpolarity Masks

In RelDTT, modes were natural numbers (minus one) expressing the number of available relations. In NatPT, we will specify for each of these relations whether it is directed or not.

**Definition 1.1.0°1.** An **anpolarity**[1] is an element of the set $\mathbb{A} := \{⤯, \bigcirc\}$, where $⤯$ stands for polar/directed and $\bigcirc$ stands for nonpolar/symmetric. We equip $\mathbb{A}$ with the partial order $\bigcirc \sqsubseteq ⤯$, corresponding to the intuition that symmetric relations are a subset of directed (i.e. potentially asymmetric) relations.

An **anpolarity mask** or just **mask** is a list $\vec{a} \in \mathsf{List}\,\mathbb{A}$ of anpolarities. We write $\mathsf{len}(\vec{a})$ for its length, and call the numbers $0, \dots, \mathsf{len}(\vec{a}) - 1$ **degrees**. We define $\sqsubseteq$ on masks of equal length pointwise.

Both for anpolarities and for masks of equal length, we denote meets (infima) with $\sqcap$ and joins (suprema) with $\sqcup$.

[1] 'An' is Latin for 'whether', as in 'Nescio an polare sit,' meaning 'I do not know whether it is polar'.

## 1.2 Jet Sets

### 1.2.1 Definitions

**Definition 1.2.1°1.** Let $\vec{a}$ be a mask. An **$\vec{a}$-jet-set** is a set $X$ equipped with $\mathsf{len}(\vec{a})$ (proof-irrelevant[2]) reflexive relations $\rightarrowtriangle_i$ where

- $0 \leq i < \mathsf{len}(\vec{a})$ is called the **degree**,
- $\rightarrowtriangle_i$ is called the **$i$-jet relation**,
- its opposite $\leftarrowtriangle_i$ is called the **opposite $i$-jet relation**,

such that

- when $a_i = \bigcirc$, then $\rightarrowtriangle_i$ is symmetric, in which case we will denote it as $\frown_i$ and call it the $i$-**edge relation** (notwithstanding that we still consider it a special case of a jet relation),
- $x \rightarrowtriangle_i y$ implies both $x \rightarrowtriangle_{i+1} y$ and $x \leftarrowtriangle_{i+1} y$ whenever $0 \leq i < i+1 < \mathsf{len}(\vec{a})$.

A **morphism** of $\vec{a}$-jet-sets is a function that preserves all the jet and edge relations.

The category of $\vec{a}$-jet-sets is called $\mathsf{JetSet}(\vec{a})$.

**Definition 1.2.1°2.** A jet set morphism is called **full** if it reflects all jet relations.

**Definition 1.2.1°3.** A jet set morphism $f : X \to Y$ is called **jet-surjective** if it is surjective as a function, and moreover for any $\vec{y} \rightarrowtriangle_j \vec{y}'$ in $Y$, there exist $\vec{x} \rightarrowtriangle_j \vec{x}'$ in $X$ such that $f(\vec{x}) = \vec{y}$ and $f(\vec{x}') = \vec{y}'$.

**Definition 1.2.1°4.** A jet set is called **transitive** if each of the $i$-jet relations is transitive (i.e. a pre-order and, if $i = \bigcirc$, an equivalence relation).

The following proposition will not really be used directly, but is a nice encouragement:

**Proposition 1.2.1°5.** Let $X$ be a transitive $\vec{a}$-jet-set and $0 \leq i < j < \mathsf{len}(\vec{a})$. Then the double category whose objects are elements of $X$, morphisms are (unique) proofs of $x \rightarrowtriangle_i y$, pro-arrows are (unique) proofs of $x \rightarrowtriangle_j y$ and squares are elements of the unit type, is a pro-arrow equipment [nLaa, Woo82, Woo85].

*Proof.* It is clearly a double category. The existence of companions and conjoints is trivial. $\square$

**Definition 1.2.1°6.** We define the

- $i$-**equijet relation** $\leftrightarrowtriangle_i$ as the symmetric interior of $\rightarrowtriangle_i$, i.e. $x \leftrightarrowtriangle_i y$ if and only if $x \rightarrowtriangle_i y$ and $x \leftarrowtriangle_i y$;
- $i$-**infrajet relation** $\rightleftarrows_i$ as the symmetric closure of $\rightarrowtriangle_i$, i.e. $x \rightleftarrows_i y$ if and only if $x \rightarrowtriangle_i y$ or $x \leftarrowtriangle_i y$.

It is immediately clear that for nonpolar degrees, the jet/edge, equijet and infrajet relations coincide. In general, we can observe that $x \rightleftarrows_i y$ implies $x \leftrightarrowtriangle_j y$ for $i < j$. So for mode $[\times\!\!\!\nearrow, \times\!\!\!\nearrow, \bigcirc]$, we get

$$\begin{tikzcd}
 & \rightarrowtriangle_0 \arrow[dr, dotted] & & & \rightarrowtriangle_1 \arrow[dr, dotted] & & & \rightarrowtriangle_2 \arrow[dl, dotted, leftrightarrow] \arrow[d, dotted, leftrightarrow] \arrow[dr, dotted, leftrightarrow] & \\
\leftrightarrowtriangle_0 \arrow[ur, dotted] \arrow[dr, dotted] & & \rightleftarrows_0 \arrow[r, dotted] & \leftrightarrowtriangle_1 \arrow[ur, dotted] \arrow[dr, dotted] & & \rightleftarrows_1 \arrow[r, dotted] & \leftrightarrowtriangle_2 \arrow[r, dotted, leftrightarrow] & \frown_2 \arrow[r, dotted, leftrightarrow] & \rightleftarrows_2 \\
 & \leftarrowtriangle_0 \arrow[ur, dotted] & & & \leftarrowtriangle_1 \arrow[ur, dotted] & & & \leftarrowtriangle_2 \arrow[ul, dotted, leftrightarrow] \arrow[u, dotted, leftrightarrow] \arrow[ur, dotted, leftrightarrow] &
\end{tikzcd} .$$

**Definition 1.2.1°7.** Let $\vec{a}$ be a mask, $i < \mathsf{len}(\vec{a})$ and $X \in \mathrm{Obj}(\mathsf{JetSet}(\vec{a}))$. We define the $i$-**opposite** $\mathsf{Op}_i(X)$ **of** $X$ as the jet set with the same carrier and relations as $X$ except that the $i$-jet relation is reversed: $x \rightarrowtriangle_i^{\mathsf{Op}_i(X)} y$ if and only if $x \leftarrowtriangle_i^X y$. This defines a functor $\mathsf{Op}_i(X) : \mathsf{JetSet}(\vec{a}) \to \mathsf{JetSet}(\vec{a})$.

[2] So these relations are functions $X \to X \to \mathsf{Prop}$ where $\mathsf{Prop}$ is a universe of $h$-propositions [Uni13]. In most applications, these relations will be decidable, but we do not require this.

We have $\mathsf{Op}_i \circ \mathsf{Op}_i = \mathrm{Id}$ and if $a_i = \bigcirc$ then $\mathsf{Op}_i = \mathrm{Id}$.

**Definition 1.2.1°8.** Write $\vec{a} \sqsubset_i \vec{b}$ if $\mathsf{len}(\vec{a}) = \mathsf{len}(\vec{b})$, $a_j = b_j$ for all $j \neq i$, $a_i = \bigcirc$ and $b_i = \times\!\!\!\!\nearrow$.

If $\vec{a} \sqsubset_i \vec{b}$, then we write $\mathsf{USym}_i : \mathsf{JetSet}(\vec{a}) \to \mathsf{JetSet}(\vec{b})$ for the forgetful functor which forgets the symmetry of $\rightarrowtriangle_i$.

**Proposition 1.2.1°9.** The forgetful functor $\mathsf{USym}_i$ is part of an adjoint triple $\mathsf{FSym}_i \dashv \mathsf{USym}_i \dashv \mathsf{CofSym}_i$ where $\mathsf{FSym}_i$ and $\mathsf{CofSym}_i$ take a $\vec{b}$-jet-set $X$ to a $\vec{a}$-jet-set of the same carrier with the same $j$-jet relations for $j \neq i$ but where

- $x \frown_i^{\mathsf{FSym}_i X} y$ if and only if $x \rightleftarrows_i^X y$,
- $x \frown_i^{\mathsf{CofSym}_i X} y$ if and only if $x \leftrightarrowtriangle_i^X y$.

We have $\mathsf{FSym}_i \circ \mathsf{USym}_i = \mathsf{CofSym}_i \circ \mathsf{USym}_i = \mathrm{Id}$, so that $\mathsf{SymCl}_i := \mathsf{USym}_i \circ \mathsf{FSym}_i$ is an idempotent monad and $\mathsf{SymInt}_i := \mathsf{USym}_i \circ \mathsf{CofSym}_i$ is an idempotent comonad.

**Definition 1.2.1°10.** We extend the definition of $\mathsf{SymCl}_i$ and $\mathsf{SymInt}_i$ to endofunctors on $\mathsf{JetSet}(\vec{b})$ where $b_i$ can be *any* anpolarity:

- If $b_i = \times\!\!\!\!\nearrow$ then they are defined as above,
- If $b_i = \bigcirc$ then they are defined as the identity functor.

Either way, they are an idempotent (co)monad and we have $\mathsf{SymCl}_i \dashv \mathsf{SymInt}_i$.

### 1.2.2 Intervals and Prisms

**Definition 1.2.2°1.** Let $\vec{a}$ be a mask and $i < \mathsf{len}(\vec{a})$.

- The $i$-**jet interval** $\llparenthesis \rightarrowtriangle_i \rrparenthesis$ is defined as the $\vec{a}$-jet-set with carrier $\{0,1\}$ and relations generated by $0 \rightarrowtriangle_i 1$.
- The **opposite** $i$-**jet interval** $\llparenthesis \leftarrowtriangle_i \rrparenthesis$ is defined as the $\vec{a}$-jet-set with carrier $\{0,1\}$ and relations generated by $0 \leftarrowtriangle_i 1$.
- The $i$-**equijet interval** $\llparenthesis \leftrightarrowtriangle_i \rrparenthesis$ is defined as the $\vec{a}$-jet-set with carrier $\{0,1\}$ and relations generated by $0 \leftrightarrowtriangle_i 1$.

If $a_i = \bigcirc$ then $\llparenthesis \rightarrowtriangle_i \rrparenthesis = \llparenthesis \leftarrowtriangle_i \rrparenthesis = \llparenthesis \leftrightarrowtriangle_i \rrparenthesis =: \llparenthesis \frown_i \rrparenthesis$ is called the $i$-**edge interval**.

Note that it would be meaningless to define an $i$-infrajet interval in the same way.

**Definition 1.2.2°2.** Let $\vec{a}$ be a mask, $i < \mathsf{len}(\vec{a})$ and $X \in \mathrm{Obj}(\mathsf{JetSet}(\vec{a}))$. We define the $i$-**twisted-prism** $X \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis$ on $X$ as the $\vec{a}$-jet-set with

- carrier $X \times \{0,1\}$,
- jet relations generated by the following requirements:
  - $(\sqcup, 0) : \mathsf{Op}_i(X) \to X \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis$ is a jet set morphism,
  - $(\sqcup, 1) : X \to X \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis$ is a jet set morphism,
  - $(x, 0) \rightarrowtriangle_i (x, 1)$ for all $x \in X$.

This defines the $i$-**twisted-prism functor** $\sqcup \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis : \mathsf{JetSet}(\vec{a}) \to \mathsf{JetSet}(\vec{a})$.

We define the **opposite** $i$-**twisted-prism** $X \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis$ on $X$ as the jet set of mask $\vec{a}$ with

- carrier $X \times \{0,1\}$,
- jet relations generated by the following requirements:
  - $(\sqcup, 0) : X \to X \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis$ is a jet set morphism,
  - $(\sqcup, 1) : \mathsf{Op}_i(X) \to X \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis$ is a jet set morphism,
  - $(x, 0) \leftarrowtriangle_i (x, 1)$ for all $x \in X$.

This defines the **opposite $i$-twisted-prism functor** $\sqcup \ltimes (\!|{\rightarrowtriangle_i}|\!) : \mathsf{JetSet}(\vec{a}) \to \mathsf{JetSet}(\vec{a})$.

If $a_i = \bigcirc$, then we call this simply the **$i$-prism functor** $\sqcup \ltimes (\!|{\frown_i}|\!)$.

Note that in both instances, we take the opposite at the source-side of the jet interval by which we multiply. This makes it unclear what a prism functor for the equijet interval $(\!|{\leftrightarrowtriangle_i}|\!)$ should look like.

**Corollary 1.2.2°3.** We have $\sqcup \ltimes (\!|{\leftarrowtriangle_i}|\!) = \mathsf{Op}_i(\sqcup \ltimes (\!|{\rightarrowtriangle_i}|\!))$. ☐

**Corollary 1.2.2°4.** Let $F_i$ be a functor between jet set categories of any of the following forms: $\mathsf{Op}_i$, $\mathsf{FSym}_i$, $\mathsf{USym}_i$, $\mathsf{CofSym}_i$, $\sqcup \ltimes (\!|{\rightarrowtriangle_i}|\!)$, $\sqcup \ltimes (\!|{\leftarrowtriangle_i}|\!)$, $\sqcup \ltimes (\!|{\frown_i}|\!)$. Let $G_j$ be a functor between jet set categories also of one of these forms, but for a different degree $j$. Then $F_i$ and $G_j$ commute, i.e. there is a natural isomorphism $F_i G_j \cong G_j F_i$.

**Corollary 1.2.2°5.** The functor $\sqcup \ltimes (\!|{\frown_i}|\!)$ commutes with itself, i.e. $(x, v, w) \mapsto (x, w, v)$ is a natural automorphism of $\sqcup \ltimes (\!|{\frown_i}|\!) \ltimes (\!|{\frown_i}|\!)$. ☐

## 1.3 Jet Topes

**Definition 1.3.0°1.** In a jet set $X$, write $\rightarrowtriangle_i^*$ for the transitive closure of the $\rightarrowtriangle_i$, and similarly $\rightleftarrows_i^*$ for the transitive closure of $\rightleftarrows_i$.

- We say that $x$ and $y$ are **$i$-connected** if $x \rightleftarrows_i^* y$, and correspondingly define **$i$-connected components**. We say that $X$ is **$i$-connected** if any two elements are $i$-connected.
- We say that $X$ is **$i$-orientable** if on any $i$-connected component, the relation $\rightarrowtriangle_i^*$ is a total pre-order, i.e. if $x \rightleftarrows_i^* y$ implies either $x \rightarrowtriangle_i^* y$ or $x \leftarrowtriangle_i^* y$.
- We say that $X$ is **$i$-acyclic** if $x \rightarrowtriangle_i^* y$ and $x \leftarrowtriangle_i^* y$ together imply $x \leftrightarrowtriangle_i^* y$.
- We say that $X$ is **$i$-antisymmetric** if $x \leftrightarrowtriangle_i y$ implies $x \rightleftarrows_{i-1} y$ for $i > 0$, or $x = y$ for $i = 0$.

Note that $i$-orientability and $i$-acyclicity are vacuous at symmetric degrees $i$. On the other hand, $i$-antisymmetry at a symmetric degree $i$ would completely define the $i$-edge relation.

**Definition 1.3.0°2.** Let $\vec{a}$ be a mask of length $n$. An $\vec{a}$-jet-set $X$ is a **jet tope** if it is:

1. finite,
2. $(n-1)$-connected if $n > 0$, or a subsingleton if $n = 0$,
3. $i$-orientable for every degree $i$,
4. $i$-acyclic for every degree $i$,
5. $i$-antisymmetric for every *directed* degree $i$.

The **category of jet topes** $\mathsf{JetTope}(\vec{a})$ is defined as the full subcategory of $\mathsf{JetSet}(\vec{a})$ on jet topes.

The conceptual reason for asking $i$-antisymmetry is to exclude the equijet interval $(\!|{\leftrightarrowtriangle_i}|\!)$, so that the only jet tope intervals at directed degrees are $(\!|{\rightarrowtriangle_i}|\!)$ and $(\!|{\leftarrowtriangle_i}|\!)$. Having an $i$-equijet interval object in the base category, will cause presheaves to contain 'formal' $i$-equijets as an additional symmetric relation, stricter than the $i$-jet relation, which we wish to avoid.

**Proposition 1.3.0°3.** Jet topes are closed under the functors $\mathsf{Op}_i$, $\mathsf{FSym}_i$, $\sqcup \ltimes (\!|{\rightarrowtriangle_i}|\!)$ and $\sqcup \ltimes (\!|{\leftarrowtriangle_i}|\!)$.

*Proof.* Clearly, each of these functors preserves finiteness.

Moreover, by existence of $0 \leq i < n$, we conclude that $n > 0$, so our jet tope needs to be $(n-1)$-connected rather than a subsingleton.

We prove the other properties in the definition of a jet tope:

$\boxed{\mathsf{Op}_i}$ Trivial.

$\boxed{\mathsf{FSym}_i}$ This functor (defined in proposition 1.2.1°9) clearly preserves $(n-1)$-connectedness even if $i = n - 1$. Meanwhile, $j$-orientability and $j$-antisymmetry are unaffected for $j \neq i$ and not required for $j = i$. In the case of $i$-antisymmetry, we use that $\rightleftarrows_{i-1}$ and equality (if $i = 0$) remain unchanged.

$\boxed{\sqcup \ltimes (\!|\rightarrowtriangle_i|\!)}$ To see $(n-1)$-connectedness, note that:

- All objects of the form $(x, 0)$ are $(n-1)$-connected,
- All objects of the form $(x, 1)$ are $(n-1)$-connected,
- Since $i \leq n-1$, we always have $(x, 0) \rightleftarrows_{n-1} (x, 1)$.

Assume $a_j = \nearrow$. For $j$-orientability, $j$-acyclicity and $j$-antisymmetry, we consider 3 situations:

$j < i$ Then for every $x$, we have $(x, 0) \not\rightleftarrows_j (x, 1)$. Every $j$-connected component $C \subseteq X$ produces two $j$-connected components $\{(c, 0) \mid c \in C\}$ and $\{(c, 1) \mid c \in C\}$, where $j$-orientability, $j$-acyclicity and $j$-antisymmetry are inherited from $X$.

$j > i$ Then for every $x$, we have $(x, 0) \leftrightarrowtriangle_j (x, 1)$. Every $j$-connected component $C \subseteq X$ produces one $j$-connected component $\{(c, u) \mid c \in C, u \in \{0, 1\}\}$, where totality of $\rightarrowtriangle_j^*$ is inherited from $X$.
For $j$-acyclicity, note that $(x, u) \rightarrowtriangle_j^* (y, v) \Leftrightarrow x \rightarrowtriangle_j^* y$ and $(x, u) \leftrightarrowtriangle_j^* (y, v) \Leftrightarrow x \leftrightarrowtriangle_j^* y$. Thus, $j$-acyclicity is also inherited from $X$.
For $j$-antisymmetry, note that $(x, u) \leftrightarrowtriangle_j (y, v)$ means that either $x \leftrightarrowtriangle_j y$ and $u = v$ or that $x = y$ and $u \neq v$. In the former case, we get $x \rightleftarrows_{j-1} y$ and hence $(x, u) \rightleftarrows_{j-1} (y, v)$. In the latter case, since $j - 1 \geq i$, we get $(x, 0) \rightarrowtriangle_{j-1} (x, 1)$, so that $(x, u) \rightleftarrows_{j-1} (y, v)$.

$j = i$ Then $(x, u) \rightleftarrows_i^* (y, v)$ if and only if $x \rightleftarrows_i^* y$. By $i$-orientability of the original jet tope $X$, we have $x \rightarrowtriangle_i^* y$ or $x \leftarrowtriangle_i^* y$. Say we have $x \rightarrowtriangle_i^* y$. Then we have $(y, 0) \rightarrowtriangle_i^* (x, 0) \rightarrowtriangle_i (x, 1) \rightarrowtriangle_i^* (y, 1)$, and thus either $(x, u) \rightarrowtriangle_i^* (y, v)$ or $(x, u) \leftarrowtriangle_i^* (y, v)$ depending on $u$ and $v$.
For $i$-acyclicity, note that if $(x, u) \rightarrowtriangle_i^* (y, v)$ and $(x, u) \leftarrowtriangle_i^* (y, v)$, then $u = v$, so this is inherited from $i$-acyclicity of $X$.
For $i$-antisymmetry, $(x, u) \leftrightarrowtriangle_i (y, v)$ implies $u = v$, so this is also inherited from $i$-antisymmetry of $X$.

$\boxed{\sqcup \ltimes (\!|\leftarrowtriangle_i|\!)}$ By an analogous argument. $\square$

## 1.4 Jet Cubes

In section 1.4.1, we define a family of *cube* categories $\mathsf{Cube}_M^{⌑}$ parametrized by a monad $M$ which defines the available operations on cube dimensions, and a choice $⌑ \in \{\square, \boxslash\}$ of whether we want our cubes to be affine (no diagonals) or cartesian.

In section 1.4.2, we define a family of *jet cube* categories $\mathsf{JetCube}_M^{⌑}(\omega, \vec{a})$ furthermore parametrized by a mask $\vec{a}$ and an orientation kit $\omega$, which determines whether jets can only be forward or also backward and/or bidirectional.

In section 1.4.3, we characterize morphisms in certain jet cube categories using a calculus.

In section 1.4.4, we shed light on jet cubes from a different corner by introducing the *semisymmetric separated product.*

In section 1.4.5 we compare our general class of jet cube categories to existing cube categories in the literature.

### 1.4.1 Cube Categories

We introduce a family of cube categories with one flavour of dimension. Fix a monad $M$ on $\mathsf{Set}$.

**Example 1.4.1°1.** Typically $M$ will be one of the following:

- The 'exception' monad $\mathsf{Pt}_2$ that sends a set $X$ to $X \uplus \{0, 1\}$, which is the carrier of the free bipointed set over $X$;
- The monad $\mathsf{IPt}_2$ that sends a set $X$ to $X \uplus \{\neg x \mid x \in X\} \uplus \{0, 1\}$, which is the carrier of the free [bipointed set equipped with an involution $\neg$ that swaps 0 and 1] over $X$;
- The monad $\mathsf{DL}$ that sends a set $X$ to the carrier of the free distributive lattice over $X$;
- The monad $\mathsf{DM}$ that sends a set $X$ to the carrier of the free De Morgan algebra over $X$;
- The monad $\mathsf{Boo}$ that sends a set $X$ to the carrier of the free boolean algebra over $X$.

#### 1.4.1 (a) Cartesian Cubes

**Definition 1.4.1°2.** We construct the **(named) category of cartesian $M$-cubes** $\mathsf{Cube}^{\boxslash}_M$ (and $\mathsf{NCube}^{\boxslash}_M$ resp.) stepwise:

- The Kleisli category $\mathsf{Kl}(M)$ of $M$ has objects $\overline{X}$ where $X$ is a set, and its morphisms $\overline{f} : \overline{X} \to \overline{Y}$ are functions $f : X \to MY$.
- Of this, we take the opposite $\mathsf{Kl}(M)^{\mathsf{op}}$. (This is the Lawvere theory corresponding to the monad $M$.)
- We define $\mathsf{NCube}^{\boxslash}_M$ as the full subcategory of $\mathsf{Kl}(M)^{\mathsf{op}}$ on finite sets. (Alternatively, this is the opposite Kleisli category of the restriction of $M$ to finite sets, either as a monad on $\mathsf{FinSet}$ or as a relative monad $\mathsf{FinSet} \to \mathsf{Set}$.)
- We define $\mathsf{Cube}^{\boxslash}_M$ as a designate skeleton of $\mathsf{NCube}_M$, e.g. the full subcategory of $\mathsf{NCube}^{\boxslash}_M$ on sets of the form $\{0, \dots, n-1\}$ with $n \geq 0$.

Objects of $\mathsf{NCube}^{\boxslash}_M$ will be denoted as tuples of names $(\mathbf{i}_0 : \mathbb{I}, \dots, \mathbf{i}_{n-1} : \mathbb{I})$ where $\mathbb{I}$ is meaningless but conveys the intuition that we regard $\mathbf{i}_k$ as a value ranging over the interval (the cube given by the singleton object). A morphism $\varphi : (\mathbf{i}_0 : \mathbb{I}, \dots, \mathbf{i}_{n-1} : \mathbb{I}) \to (\mathbf{j}_0 : \mathbb{I}, \dots, \mathbf{j}_{m-1} : \mathbb{I})$ is then a function sending each $\mathbf{j}_k$ to an expression $\mathbf{j}_k\langle\varphi\rangle \in M\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}$. The morphism $\varphi$ will also be denoted as $(\mathbf{j}_0\langle\varphi\rangle/\mathbf{j}_0, \dots, \mathbf{j}_{m-1}\langle\varphi\rangle/\mathbf{j}_{m-1})$. The situation in $\mathsf{Cube}^{\boxslash}_M$ is the same except that we now regard the names $\mathbf{i}_k$ as De Bruijn indices.

**Corollary 1.4.1°3.** The categories $\mathsf{Cube}^{\boxslash}_M$ and $\mathsf{NCube}^{\boxslash}_M$ have finite products, given by finite coproducts of sets. $\square$

#### 1.4.1 (b) Affine Cubes

If $T$ is a *container* monad [Uus17], i.e. a monad whose underlying functor is a container functor [AAG05] of the form $TX = \Sigma(s : S).(P(s) \to X)$, then we define $T^{\#}X$ as the set of *affine* expressions $\Sigma(s : S).(P(s) \hookrightarrow X)$, which is an endofunctor on the category $\mathsf{Set}^{\hookrightarrow}$ of sets and injective functions. If $M$ is merely a *quotient* of a container monad, i.e. $M$ is of the form $MX = TX/\sim_X$ with $T$ as above, then we define $M^{\#}X$ as the set of equivalence classes with an affine representant.

**Remark 1.4.1°4.** An important source of monads such as $M$ are monads specified by a syntactic algebraic theory [Man12, ARVL10, Nuy22]. A syntactic algebraic theory specifies a set of operations $S_0$, assigns to each operation $s : S_0$ an arity $P_0(s)$ which is again a set, and subjects these to a set of axioms.[3] The container $(S_0, P_0)$ specifies a container functor $FX = \Sigma(s : S_0).(P_0(s) \to X)$ on $\mathsf{Set}$. A free monad $F^*$ over this functor $F$ exists and satisfies the fixpoint equation $F^*X \cong X \uplus FF^*X$. We remark that the free monad $F^*$ over a container functor $F$ is again a container functor, i.e. there exists a container $(S, P)$ such that $F^*X = \Sigma(s : S).(P(s) \to X)$ specifies the free monad over $F$. The axioms determine an equivalence relation $\sim_X$ on $F^*X$ such that $MX := F^*X/\sim_X$ is again a monad. This situation applies to each of the monads in example 1.4.1°1.

In fact, often the quotient can be taken already at the level of the container, so that there exists a container $(S', P')$ such that $MX \cong \Sigma(s : S').(P'(s) \to X)$.

We say that $(s, f), (s', f') \in T^{\#}X$ are **mutually fresh**, denoted $(s, f) \# (s', f')$, if the images of $f$ and $f'$ are disjoint. Elements of $M^{\#}X$ are mutually fresh if they have mutually fresh representants. We call the monad $(T, \eta, \mu)$ **affine** if $\eta_X : X \to TX$ lands in $T^{\#}X$ for all $X$ and $\mu_X : TTX \to TX$ restricted to $(TT)^{\#}X$ (note that container functors are closed under composition) lands in $T^{\#}X$; and similar for $M$.

**Definition 1.4.1°5.** Let $M$ be a quotient of a container monad, and let it be affine. We construct the **(named) category of affine $M$-cubes** $\mathsf{Cube}^{\square}_M$ (and $\mathsf{NCube}^{\square}_M$ resp.) stepwise:

[3] We use 'syntactic algebraic theory' to refer to the syntactic presentation as described here, and 'monad' and 'Lawvere theory' to refer to the less syntactic objects they specify.

- The affine Kleisli category $\mathsf{Kl}^{\#}(M)$ has objects $\overline{X}$ where $X$ is a set, and its morphisms $\overline{f} : \overline{X} \to \overline{Y}$ are functions $f : X \to M^{\#}Y$ such that for any $x \neq x'$ in $X$, we have $f(x) \# f(x')$. Identity and composition are well-defined because $M$ is affine.
- Of this, we take the opposite $\mathsf{Kl}^{\#}(M)^{\mathsf{op}}$.
- We define $\mathsf{NCube}^{\square}_{M}$ as the full subcategory of $\mathsf{Kl}^{\#}(M)^{\mathsf{op}}$ on finite sets.
- We define $\mathsf{Cube}^{\square}_{M}$ as a designate skeleton of $\mathsf{NCube}^{\square}_{M}$, e.g. the full subcategory of $\mathsf{NCube}^{\square}_{M}$ on sets of the form $\{0, \dots, n-1\}$ with $n \geq 0$.

Objects will be represented as for the cartesian cube categories.

**Corollary 1.4.1°6.** The categories $\mathsf{Cube}^{\square}_{M}$ and $\mathsf{NCube}^{\square}_{M}$ have a symmetric monoidal structure $(\top, *)$ given by finite coproducts of sets. The binary operation is called the **separated product**. $\square$

### 1.4.1 (c) Examples

This way, we get – among others – the following cube categories:

$\mathsf{Cube}^{\boxslash}_{\mathsf{Pt}_2}$ The cartesian cube category. A morphism $\varphi : V \to W$ sends every dimension $\mathbf{j} \in W$ to $\mathbf{j}\langle\varphi\rangle \in V \cup \{0,1\}$. Its cubes have diagonals.

$\mathsf{Cube}^{\square}_{\mathsf{Pt}_2}$ The affine cube category [BCH14]. A morphism $\varphi : V \to W$ sends every dimension $\mathbf{j} \in W$ to $\mathbf{j}\langle\varphi\rangle \in V \cup \{0,1\}$, such that if $\mathbf{j}\langle\varphi\rangle = \mathbf{j}'\langle\varphi\rangle \in V$ then $\mathbf{j} = \mathbf{j}'$. Its cubes have no diagonals.

$\mathsf{Cube}^{\boxslash}_{\mathsf{IPt}_2}$ The symmetric cartesian cube category. We have a negation/involution/symmetry $(\neg\mathbf{i}/\mathbf{j}) : (\mathbf{i} : \mathbb{I}) \to (\mathbf{j} : \mathbb{I})$.

$\mathsf{Cube}^{\boxslash}_{\mathsf{DL}}$ The cartesian cube category with connections. We have morphisms $(\mathbf{i} \vee \mathbf{j}/\mathbf{k}), (\mathbf{i} \wedge \mathbf{j}/\mathbf{k}) : (\mathbf{i} : \mathbb{I}, \mathbf{j} : \mathbb{I}) \to (\mathbf{k} : \mathbb{I})$. There are no symmetries

$\mathsf{Cube}^{\boxslash}_{\mathsf{DM}}$ The CCHM cube category [CCHM15], which combines symmetries and connections. We have $(\mathbf{i} \wedge \neg\mathbf{i}/\mathbf{j}) \neq (0/\mathbf{j}) : (\mathbf{i} : \mathbb{I}) \to (\mathbf{j} : \mathbb{I})$ and $(\mathbf{i} \vee \neg\mathbf{i}/\mathbf{j}) \neq (1/\mathbf{j}) : (\mathbf{i} : \mathbb{I}) \to (\mathbf{j} : \mathbb{I})$.

$\mathsf{Cube}^{\boxslash}_{\mathsf{Boo}}$ A cube category very similar to the CCHM one, but we have $(\mathbf{i} \wedge \neg\mathbf{i}/\mathbf{j}) = (0/\mathbf{j}) : (\mathbf{i} : \mathbb{I}) \to (\mathbf{j} : \mathbb{I})$ and $(\mathbf{i} \vee \neg\mathbf{i}/\mathbf{j}) = (1/\mathbf{j}) : (\mathbf{i} : \mathbb{I}) \to (\mathbf{j} : \mathbb{I})$.

We remark that $\mathsf{Cube}^{\square}_{\mathsf{DM}}$ and $\mathsf{Cube}^{\square}_{\mathsf{Boo}}$ should be isomorphic as the additional law of boolean algebras w.r.t. de Morgan algebras only affects non-affine expressions.

### 1.4.1 (d) The Endpoint Model

We remarked above that $\mathcal{L} := \mathsf{Kl}(M)^{\mathsf{op}}$ is the Lawvere category of $M$. It is known then (see e.g. [Nuy22]), that the Eilenberg-Moore category of $M$ (which is the category of Eilenberg-Moore algebras of $M$) is equivalent category of models of $\mathcal{L}$ (which is the category of product-preserving functors $\mathcal{L} \to \mathsf{Set}$). Such functors are fully determined by the image of the singleton set (as every set is a coproduct of singletons and the Kleisli-category retains coproducts) and that image will be exactly the carrier of the corresponding Eilenberg-Moore algebra.

It is clear that both the cartesian and affine (named) cube categories are subcategories of $\mathcal{L}$. As such, any $M$-algebra induces a functor $\mathcal{L} \to \mathsf{Set}$ and hence a functor from either of the $M$-cube categories to $\mathsf{Set}$.

The initial algebra of any monad $M$ on $\mathsf{Set}$ has carrier $M\varnothing$, which for each of the monads in example 1.4.1°1 equals $\{0,1\}$. Correspondingly, the initial model of $\mathcal{L}$ is the functor $\mathsf{EP} : \mathcal{L} \to \mathsf{Set}$ sending $(\mathbf{i}_0 : \mathbb{I}, \dots, \mathbf{i}_{n-1} : \mathbb{I})$ to $\{0,1\}^{\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}}$. We call this the **endpoint model**. It is naturally isomorphic (in fact equal) to $\mathrm{Hom}_{\mathcal{L}}((), \sqcup) : \mathcal{L} \to \mathsf{Set}$, since we have

$$\mathrm{Hom}_{\mathcal{L}}((), (\mathbf{i}_0 : \mathbb{I}, \dots, \mathbf{i}_{n-1} : \mathbb{I})) = \mathrm{Hom}_{\mathsf{Kl}(M)}(\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}, \varnothing) = (M\varnothing)^{\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}} = \{0,1\}^{\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}}.$$

Recall that a morphism $\varphi : (\mathbf{i}_0 : \mathbb{I}, \dots, \mathbf{i}_{n-1} : \mathbb{I}) \to (\mathbf{j}_0 : \mathbb{I}, \dots, \mathbf{j}_{m-1} : \mathbb{I})$ assigns to each $\mathbf{j}$ a value $\mathbf{j}\langle\varphi\rangle$ in $M\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}$, the free $M$-algebra over $\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}$. The function $\mathsf{EP}(\varphi)$ is defined by

$$\mathsf{EP}(\varphi)\left(v^{\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\} \to \{0,1\}}\right)(\mathbf{j}) = \alpha(M(v)(\mathbf{j}\langle\varphi\rangle)),$$

where $\alpha : M\{0,1\} \to \{0,1\}$ is the algebra structure on $\{0,1\}$. Using the operation $\ggg=^{\alpha} : MX \to (X \to \{0,1\}) \to \{0,1\} : \hat{x} \mapsto f \mapsto \alpha(Mf(\hat{x}))$, we can write this as $\mathsf{EP}(\varphi)(v)(\mathbf{j}) = \mathbf{j}\langle\varphi\rangle \ggg=^{\alpha} v$.

**Proposition 1.4.1°7.** The functor $\mathsf{EP} : \mathsf{Cube}^{\boxslash}_{\mathsf{Boo}} \to \mathsf{Set}$ is fully faithful.

*Proof.* We need to show that any function $f : \{0,1\}^{\{\mathbf{i}_0,\dots,\mathbf{i}_{n-1}\}} \to \{0,1\}^{\{\mathbf{j}_0,\dots,\mathbf{j}_{m-1}\}}$ can be obtained as some $\mathsf{EP}(\varphi)$ with $\varphi : (\mathbf{i}_0 : \mathbb{I}, \dots, \mathbf{i}_{n-1} : \mathbb{I}) \to (\mathbf{j}_0 : \mathbb{I}, \dots, \mathbf{j}_{m-1} : \mathbb{I})$. We remark that such a function $f$ in fact consists of $m$ truth tables in $n$ boolean variables. From the full disjunctive normal form, it is clear that elements of the free boolean algebra are in 1-1 correspondence with truth tables. Concretely, for each $\mathbf{j}$, define $\mathbf{j}\langle\varphi\rangle$ to be the element of $\mathsf{Boo}\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}$ corresponding to the truth table $f(\sqcup, \mathbf{j})$. Then $\mathbf{j}\langle\varphi\rangle \ggg=^{\alpha} v$ will evaluate $\mathbf{j}\langle\varphi\rangle$ after replacing each variable $\mathbf{i}$ with its value $v(\mathbf{i})$, yielding the value $f(v, \mathbf{j})$ prescribed by the truth table $f(\sqcup, \mathbf{j})$. □

**Proposition 1.4.1°8.** The obvious functor $I : \mathsf{Cube}^{\maltese}_{M} \to \mathsf{Cube}^{\sqlozenge}_{N}$ where

- $\maltese, \sqlozenge \in \{\Box, \boxslash\}$ and $\maltese \leq \sqlozenge$ according to the order $\Box \leq \boxslash$,
- $M, N \in \{\mathsf{Pt}_2, \mathsf{IPt}_2, \mathsf{DL}, \mathsf{Boo}\}$ and $M \leq N$ according to the partial order

$$\begin{tikzcd}
\mathsf{Pt}_2 \arrow[r, "\leq"] \arrow[d, "\rotatebox{90}{$\leq$}"'] & \mathsf{IPt}_2 \arrow[d, "\rotatebox{90}{$\leq$}"'] \arrow[ddr, bend left, "\rotatebox{90}{$\leq$}"] & \\
\mathsf{DL} \arrow[r, "\leq"] \arrow[drr, bend right, "\leq"'] & \mathsf{DM} & \\
 & & \mathsf{Boo},
\end{tikzcd}$$

is faithful.

*Proof.* In a first step, it is obvious by construction that $\mathsf{Cube}^{\Box}_{M} \to \mathsf{Cube}^{\boxslash}_{M}$ is faithful.

In a second step, note that we have a monad morphism $\iota : M \to N$ such that $\iota_X : M(X) \to N(X)$ is injective for all $X$. Then the resulting functor between the Kleisli categories, which are opposite to the cartesian cube categories, is faithful. □

Clearly, the functor $I : \mathsf{Cube}^{\boxslash}_{\mathsf{DM}} \to \mathsf{Cube}^{\boxslash}_{\mathsf{Boo}}$ is not faithful: it sends the morphisms $(0/\mathbf{j}), (\mathbf{i} \wedge \neg\mathbf{i}/\mathbf{j}) : (\mathbf{i} : \mathbb{I}) \to (\mathbf{j} : \mathbb{I})$ in $\mathsf{Cube}^{\boxslash}_{\mathsf{DM}}$ to the same morphism in $\mathsf{Cube}^{\boxslash}_{\mathsf{Boo}}$.

**Corollary 1.4.1°9.** The functor $\mathsf{EP} : \mathsf{Cube}^{\sqlozenge}_{M} \to \mathsf{Set}$ is faithful for each $M \in \{\mathsf{Pt}_2, \mathsf{IPt}_2, \mathsf{DL}, \mathsf{Boo}\}$.

*Proof.* Follows by composing proposition 1.4.1°7 and proposition 1.4.1°8. □

### 1.4.2 Jet Cubes

#### 1.4.2 (a) Jet Cube Objects

**Definition 1.4.2°1.** An **orientation kit** is one of the following sets of formal symbols:[4]

$$\mathsf{f} = \{\rightarrowtriangle\}, \qquad \mathsf{fe} = \{\rightarrowtriangle, \leftrightarrowtriangle\}, \qquad \mathsf{fb} = \{\rightarrowtriangle, \leftarrowtriangle\}, \qquad \mathsf{fbe} = \{\rightarrowtriangle, \leftarrowtriangle, \leftrightarrowtriangle\}.$$

The set of orientation kits will be denoted $\mathsf{OKit} = \{\mathsf{f}, \mathsf{fe}, \mathsf{fb}, \mathsf{fbe}\}$.

**Definition 1.4.2°2.** Let $\vec{a}$ be a mask and $\omega$ an orientation kit. We define the set of $(\omega, \vec{a})$**-jet-cubes** as the set of lists of elements of $\{P_i \mid P \in \omega, 0 \leq i < \mathsf{len}(\vec{a})\}$, where we identify all $P_i = Q_i =: \frown_i$ $(P, Q \in \omega)$ if $a_i = \bigcirc$. We denote jet cubes as $(\mathbf{i}_0 : \llparenthesis (P_0)_{i_0} \rrparenthesis, \dots, \mathbf{i}_{n-1} : \llparenthesis (P_{n-1})_{i_{n-1}} \rrparenthesis)$, thinking of the names $\mathbf{i}_k$ as De Bruijn indices.

**Definition 1.4.2°3.** We call a variable $\mathbf{i}$ of an $(\omega, \vec{a})$-jet-cube $i$-**symmetric** (for a degree $0 \leq i < \mathsf{len}(\vec{a})$) if *any* of the following conditions holds:

- $\mathbf{i}$ is not of degree $i$,

[4] The letters stand for *forward*, *backward* and *equijet*.

- $\mathbf{i}$ is an equijet variable, i.e. $\mathbf{i} : \llparenthesis \leftrightarrowtriangle_j \rrparenthesis$,
- $a_i = \bigcirc$.

Otherwise, it is called $i$-**directed**. Thus, if $\mathbf{i}$ is $i$-directed, then $a_i = \nearrow$ and $\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis$ or $\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis$.

**Definition 1.4.2°4.** Let $\vec{a} \sqsubset_i \vec{b}$ and $\leftrightarrowtriangle \in \omega$. For any $(\omega, \vec{a})$-jet-cube $W$, we define the $(\omega, \vec{b})$-jet-cube $\mathsf{USym}_i^{\boxdot} W$ by replacing every occurrence of $\frown_i$ with $\leftrightarrowtriangle_i$.

Note that a $\vec{b}$-jet-cube is uniquely in the image of $\mathsf{USym}_i^{\boxdot}$ if it does not feature the symbols $\rightarrowtriangle_i$ and $\leftarrowtriangle_i$, i.e. if all variables are $i$-symmetric.

**Definition 1.4.2°5.** For any $(\omega, \vec{a})$-jet-cube $W$, we define the $\vec{a}$-jet-set $\mathsf{JEP}(W)$ as follows:

$$
\begin{array}{lcll}
\mathsf{JEP}(()) & = & \top, & \\
\mathsf{JEP}(W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) & = & \mathsf{JEP}(W) \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis, & \\
\mathsf{JEP}(W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis) & = & \mathsf{JEP}(W) \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis, & \\
\mathsf{JEP}(W, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis) & = & \mathsf{JEP}(W) \ltimes \llparenthesis \frown_i \rrparenthesis, & \\
\mathsf{JEP}(W, \mathbf{i} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis) & = & \mathsf{USym}_i\, \mathsf{JEP}\big((\mathsf{USym}_i^{\boxdot})^{-1}(W, \mathbf{i} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis)\big) & \text{if } a_i = \nearrow \\
 & = & \mathsf{USym}_i\, \mathsf{JEP}\big((\mathsf{USym}_i^{\boxdot})^{-1}(W), \mathbf{i} : \llparenthesis \frown_i \rrparenthesis\big) & \\
 & = & \mathsf{USym}_i \big(\mathsf{JEP}\big((\mathsf{USym}_i^{\boxdot})^{-1}(W)\big) \ltimes \llparenthesis \frown_i \rrparenthesis\big). &
\end{array}
$$

Setting $V = (\mathsf{USym}_i^{\boxdot})^{-1}(W)$, the last equation can be rephrased as

$$
\begin{array}{lcl}
\mathsf{JEP}(\mathsf{USym}_i^{\boxdot} V, \mathbf{i} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis) = \mathsf{JEP}(\mathsf{USym}_i^{\boxdot}(V, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis)) & = & \mathsf{USym}_i(\mathsf{JEP}(V, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis)) \\
 & = & \mathsf{USym}_i(\mathsf{JEP}(V) \ltimes \llparenthesis \frown_i \rrparenthesis).
\end{array}
$$

**Definition 1.4.2°6.** We define the **jet-erasure function** $\lfloor \sqcup \rfloor$, which sends $(\omega, \vec{a})$-jet-cubes to cubes (i.e. objects of any of the cube categories defined in section 1.4.1), by

$$
\lfloor () \rfloor = (), \qquad \lfloor (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \rfloor = \lfloor (W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis) \rfloor = \lfloor (W, \mathbf{i} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis) \rfloor = \lfloor (W, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis) \rfloor = (\lfloor W \rfloor, i : \mathbb{I}).
$$

**Corollary 1.4.2°7.** For any $(\omega, \vec{a})$-jet-cube $W$, the carrier of $\mathsf{JEP}(W)$ is $\mathsf{EP}(\lfloor W \rfloor)$. Thus, every jet cube determines an object of the following strict pullback of categories:

$$
\begin{tikzcd}
\{(\omega, \vec{a})\text{-jet-cubes}\} \arrow[dr, dotted] \arrow[drr, bend left, "\mathsf{JEP}"] \arrow[ddr, bend right, "\lfloor \sqcup \rfloor"'] & & \\
 & \mathsf{Cube}_M^{\between} \times_{\mathsf{Set}} \mathsf{JetSet}(\vec{a}) \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{JetSet}(\vec{a}) \arrow[d, "U"] \\
 & \mathsf{Cube}_M^{\between} \arrow[r, "\mathsf{EP}"'] & \mathsf{Set},
\end{tikzcd}
$$

It is straightforward to see that the function thus obtained is injective. $\square$

#### 1.4.2 (b) Jet Cube Categories

**Definition 1.4.2°8.** Let $\vec{a}$ be a mask, $\omega \in \mathsf{OKit}$, $\between \in \{\square, \boxslash\}$ and $M$ a monad on $\mathsf{Set}$. We define the category $\mathsf{JetCube}_M^{\between}(\omega, \vec{a})$ of $(\omega, \vec{a})$-**jet-$M$-cubes** as the full subcategory of $\mathsf{Cube}_M^{\between} \times_{\mathsf{Set}} \mathsf{JetSet}(\vec{a})$ on $(\omega, \vec{a})$-jet-cubes, as justified by corollary 1.4.2°7. The functions $\mathsf{JEP}$ and $\lfloor \sqcup \rfloor$ are correspondingly extended to functors.

**Corollary 1.4.2°9.** For $\omega \in \{\mathsf{f}, \mathsf{fb}\}$, the functor $\mathsf{JEP} : \mathsf{JetCube}_M^{\between}(\omega, \vec{a}) \to \mathsf{JetSet}(\vec{a})$ factors over the inclusion $\mathsf{JetTope}(\vec{a}) \hookrightarrow \mathsf{JetSet}(\vec{a})$.

*Proof.* By induction on the dimension and using proposition 1.3.0°3, it is clear that for any jet cube $W$, the jet set $\mathsf{JEP}(W)$ is a jet tope, which factors the action on objects. The action on morphisms factors because $\mathsf{JetTope}(\vec{a})$ is a full subcategory of $\mathsf{JetSet}(\vec{a})$. $\square$

We will ultimately only be interested in $(\mathsf{f}, \vec{a})$-jet-cubes, but in section 1.4.3, we define an inductive predicate to determine whether a cube morphism $\varphi : \lfloor V \rfloor \to \lfloor W \rfloor$ is in fact a jet cube morphism $V \to W$, and this predicate's inference rules make use of $(\mathsf{fbe}, \vec{a})$-jet-cubes in their premises. The orientation kits fe and fb are only introduced for explanatory purposes: we can easily relate jet cubes with and without $\leftarrowtriangle$ (proposition 1.4.2°10), and later on[5] we will be able to relate jet cubes with and without $\leftrightarrowtriangle$ by inserting additional symmetric degrees.

**Proposition 1.4.2°10.** Let $(\omega, \omega') \in \{(\mathsf{f}, \mathsf{fb}), (\mathsf{fe}, \mathsf{fbe})\}$ and $M \in \{\mathsf{IPt}_2, \mathsf{Boo}\}$. Then the inclusion

$$\mathsf{JetCube}^{\square}_M(\omega, \vec{a}) \hookrightarrow \mathsf{JetCube}^{\square}_M(\omega', \vec{a}),$$

which is fully faithful by definition, is also split essentially surjective and therefore an equivalence.

*Proof.* One proves, by induction on the length of the jet cube, that any jet cube is isomorphic to the jet cube in which every occurrence of $\llparenthesis \leftarrowtriangle_i \rrparenthesis$ is replaced with $\llparenthesis \rightarrowtriangle_i \rrparenthesis$. ☐

**Proposition 1.4.2°11.** The following functors on $\vec{a}$-jet-sets lift over $\mathsf{JEP}$ to functors on $(\omega, \vec{a})$-jet-cubes under the following conditions:

$$\begin{array}{cccll}
\mathsf{Op}_i & \text{lifts to} & \mathsf{Op}_i^{\boxdot} & \text{if} & \leftarrowtriangle \in \omega \\
\mathsf{FSym}_i & \text{lifts to} & \mathsf{FSym}_i^{\boxdot} & & \\
\mathsf{USym}_i & \text{lifts to} & \mathsf{USym}_i^{\boxdot} & \text{if} & \leftrightarrowtriangle \in \omega \\
\mathsf{SymCl}_i & \text{lifts to} & \mathsf{SymCl}_i^{\boxdot} & \text{if} & \leftrightarrowtriangle \in \omega \\
\sqcup \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis & \text{lifts to} & \sqcup \ltimes (\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) & & \\
\sqcup \ltimes \llparenthesis \frown_i \rrparenthesis & \text{lifts to} & \sqcup \ltimes (\mathbf{i} : \llparenthesis \frown_i \rrparenthesis) & \text{if} & a_i = \bigcirc \\
\sqcup \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis & \text{lifts to} & \sqcup \ltimes (\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis) & \text{if} & \leftarrowtriangle \in \omega
\end{array}$$

We have $\mathsf{FSym}_i^{\boxdot} \dashv \mathsf{USym}_i^{\boxdot}$ and thus an idempotent monad $\mathsf{SymCl}_i^{\boxdot} := \mathsf{USym}_i^{\boxdot} \circ \mathsf{FSym}_i^{\boxdot}$, whose definition we extend to masks $\vec{b}$ where $b_i = \bigcirc$ as in definition 1.2.1°10.

*Proof.* The functors $\mathsf{FSym}_i$, $\mathsf{Op}_i$ and $\mathsf{USym}_i$ have no effect on the carrier, so they certainly lift to $\mathsf{Cube}^{\square}_M$, hence to the pullback $\mathsf{Cube}^{\square}_M \times_{\mathsf{Set}} \mathsf{JetSet}(\vec{a})$.

- $\mathsf{FSym}_i$ lifts to jet cubes by replacing every occurrence of $\rightarrowtriangle_i$, $\leftarrowtriangle_i$ or $\leftrightarrowtriangle_i$ with $\frown_i$.
- $\mathsf{Op}_i$ lifts to jet cubes if $\leftarrowtriangle \in \omega$ by reversing the *last* occurrence of either $\rightarrowtriangle_i$ or $\leftarrowtriangle_i$ (corollary 1.2.2°3), if present.
- $\mathsf{USym}_i$ lifts to jet cubes as the operation $\mathsf{USym}_i^{\boxdot}$ already introduced in definition 1.4.2°4.

To prove $\mathsf{FSym}^{\boxdot} \dashv \mathsf{USym}^{\boxdot}$, we need to build unit and co-unit natural transformations. Since the categories of jet cubes are fully faithful subcategories of the pullback $\mathsf{Cube}^{\square}_M \times_{\mathsf{Set}} \mathsf{JetSet}(\vec{a})$, it suffices to build them there. They were already established in $\mathsf{JetSet}(\vec{a})$ by proposition 1.2.1°9. As they reduce to the identity unit and co-unit of $\mathrm{Id} \dashv \mathrm{Id}$ for the carriers, they trivially lift to $\mathsf{Cube}^{\square}_M$.

The various prism functors multiply the carrier with $\{0, 1\}$ and thus lift over $\mathsf{EP}$ to the affine/cartesian cube category by multiplying with $(\mathbf{i} : \mathbb{I})$. Hence, they also lift to the pullback $\mathsf{Cube}^{\square}_M \times_{\mathsf{Set}} \mathsf{JetSet}(\vec{a})$. Each of them lifts to jet cubes by appending the symbol concerned (if available). ☐

**Proposition 1.4.2°12.** Any two functors on jet cubes concerned in proposition 1.4.2°11, instantiated on different degrees, commute. In other words, the natural transformation given in corollary 1.2.2°4 lifts to jet cubes when the associated functors lift.

*Proof.* Since the categories of jet cubes are fully faithful subcategories of the pullback $\mathsf{Cube}^{\square}_M \times_{\mathsf{Set}} \mathsf{JetSet}(\vec{a})$, it suffices to prove the natural isomorphism there. The isomorphism was already established in $\mathsf{JetSet}(\vec{a})$ by corollary 1.2.2°4, and the effect on the carrier is either nothing (when at most one prism functor is involved) or swapping components (when both functors are prism functors). These isomorphisms lift to $\mathsf{Cube}^{\square}_M$. ☐

[5] Refer

**Proposition 1.4.2°13.** The functor $\sqcup \ltimes (\mathbf{i} : \llparenthesis \frown_i \rrparenthesis)$ commutes with itself, i.e. the natural automorphism given in corollary 1.2.2°5 lifts to jet cubes as $(\mathbf{i}/\mathbf{i}, \mathbf{j}/\mathbf{j}) : \sqcup \ltimes (\mathbf{i} : \llparenthesis \frown_i \rrparenthesis) \ltimes (\mathbf{j} : \llparenthesis \frown_i \rrparenthesis) \cong \sqcup \ltimes (\mathbf{j} : \llparenthesis \frown_i \rrparenthesis) \ltimes (\mathbf{i} : \llparenthesis \frown_i \rrparenthesis)$.

*Proof.* Analogous to the proof of proposition 1.4.2°12. The isomorphism was already established in $\mathsf{JetSet}(\vec{a})$ by corollary 1.2.2°5, and the effect on the carrier is swapping components, which lifts to $\mathsf{Cube}^{\boxbox}_M$. $\square$

**Remark 1.4.2°14.** As of this point we will only be interested in the monads $\mathsf{IPt}_2$ and $\mathsf{Boo}$ because:

- We need involutions in order to be able to work with the source-side of the twisted prism, ruling out $\mathsf{Pt}_2$ and $\mathsf{DL}$.
- We do not see any advantage of $\mathsf{DM}$ over $\mathsf{Boo}$. In particular, we want $\mathsf{EP}$ to be faithful (proposition 1.4.1°8).

**Theorem 1.4.2°15.** Assuming decidability of the affineness predicate on cube morphisms, then for $M \in \{\mathsf{Pt}_2, \mathsf{IPt}_2\}$, we have isomorphisms of categories[6]

$$\mathsf{JetCube}^{\square}_{M}(\omega, \vec{a}) \cong \mathsf{JetCube}^{\boxslash}_{M}(\omega, \vec{a}),$$

which act as the identity on objects.

*Proof.* We know that $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\omega, \vec{a})$ is a subcategory of $\mathsf{JetCube}^{\boxslash}_{\mathsf{IPt}_2}(\omega, \vec{a})$. So we need to show that any morphism in $\mathsf{JetCube}^{\boxslash}_{\mathsf{IPt}_2}(\mathsf{fbe}, \vec{a})$ is in fact affine. Take such a morphism $\hat{\varphi} : V \to W$ (write $\varphi = \lfloor \hat{\varphi} \rfloor$) and assume it is not affine. Since $M$ only has nullary and unary operations, this means that $W$ has dimensions $\mathbf{i}$ and $\mathbf{j}$ such that $\mathbf{i}\langle\varphi\rangle$ and $\mathbf{j}\langle\varphi\rangle$ are not mutually fresh, meaning that $V$ has some dimension $\mathbf{k}$ such that $\mathbf{i}\langle\varphi\rangle, \mathbf{j}\langle\varphi\rangle \in \{\mathbf{k}, \neg\mathbf{k}\}$. Then $\mathsf{JEP}(\hat{\varphi})$ cannot be a jet set morphism as $\mathsf{JEP}(W)$ has no diagonals. This is a contradiction. $\square$

Note that the situation is not so simple for $\mathsf{Boo}$. For example, at symmetric degrees, $\mathsf{JetCube}^{\boxslash}_{\mathsf{Boo}}(\mathsf{f}, \vec{a})$ features the 'exclusive or' operation

$$((\mathbf{i} \vee \mathbf{j}) \wedge \neg(\mathbf{i} \wedge \mathbf{j})/\mathbf{k}) : (\mathbf{i} : \llparenthesis \frown_i \rrparenthesis, \mathbf{j} : \llparenthesis \frown_i \rrparenthesis) \to (\mathbf{k} : \llparenthesis \frown_i \rrparenthesis)$$

which cannot be constructed in $\mathsf{JetCube}^{\square}_{\mathsf{Boo}}(\mathsf{f}, \vec{a})$. More startlingly, even at directed degrees, we have operations such as the following:

$$(\mathbf{i} \wedge \mathbf{j}/\mathbf{p}, \mathbf{j} \wedge \mathbf{k}/\mathbf{q}) : (\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis, \mathbf{j} : \llparenthesis \rightarrowtriangle_i \rrparenthesis, \mathbf{k} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \to (\mathbf{p} : \llparenthesis \rightarrowtriangle_i \rrparenthesis, \mathbf{q} : \llparenthesis \rightarrowtriangle_i \rrparenthesis),$$

[6] Depending on the formalization, possibly even equalities.

which collapses five consecutive points of the Hamiltonian path and is a legitimate jet cube morphism:

$$\begin{tikzcd}
(1,0,0) \arrow[rrr] & & & (1,0,1) \arrow[dl] \arrow[ddd, dotted] \\
& (0,0,0) \arrow[ul] \arrow[r, dotted] & (0,0,1) \arrow[d] & \\
& (0,1,0) \arrow[u] \arrow[r, dotted] & (0,1,1) \arrow[dr] & \\
(1,1,0) \arrow[uuu, dotted] \arrow[ur] \arrow[rrr, dotted] & & & (1,1,1)
\end{tikzcd}$$

↧

$$\begin{tikzcd}
(0,0) \arrow[rrr, equal] & & & (0,0) \\
& (0,0) \arrow[ul, equal] \arrow[r, dotted, equal] \arrow[d, equal] & (0,0) \arrow[ur, equal] \arrow[d] & \\
& (0,0) \arrow[r, dotted] & (0,1) \arrow[dr] & \\
(1,0) \arrow[uuu, dotted] \arrow[ur] \arrow[rrr, dotted] & & & (1,1).
\arrow[from=1-4, to=4-4, dotted]
\end{tikzcd}$$

While the operations $\vee$ and $\wedge$ in themselves are useful in developing a base category for pro-arrow equipments in order to extract companion and conjoint squares from an arrow, cube transformations such as the one above are not assumed in the definition of pro-arrow equipments, so we wish to exclude these. For this reason, we will no longer be interested in cartesian jet-Boo-cubes. By theorem 1.4.2°15, we are also no longer interested in cartesian jet-$\mathsf{IPt}_2$-cubes. In short then, by remark 1.4.2°14:

**Remark 1.4.2°16.** We are no longer interested in cartesian jet cubes.

### 1.4.3 A Calculus for Jet Cube Morphisms

In this section, we develop a calculus that inductively generates the morphisms of the category $\mathsf{JetCube}^{\square}_{M}(\mathsf{fbe}, \vec{a})$ and therefore also those of its full subcategories $\mathsf{JetCube}^{\square}_{M}(\omega, \vec{a})$.

Since the forgetful functor $U : \mathsf{JetSet}(\vec{a}) \to \mathsf{Set}$ is faithful, so is $\lfloor \sqcup \rfloor : \mathsf{JetCube}^{\square}_{M}(\omega, \vec{a}) \to \mathsf{Cube}^{\square}_{M}$. As such, we can regard 'being a morphism of jet cubes' as a proof-irrelevant property of morphisms of cubes, which we will therefore use as preterms. Our calculus will therefore feature a single judgement $\vdash \varphi : V \to W$ meaning that the morphism $\varphi : \lfloor V \rfloor \to \lfloor W \rfloor$ is in fact a morphism of jet cubes. Soundness (theorem 1.4.3°4) of the calculus will be the property that the judgement's meaning actually holds when the judgement is derivable, whereas completeness (theorem 1.4.3°26) means that the judgement is derivable when its meaning is true. We do not have to bother with an equational theory, as we can simply inherit it from $\mathsf{Cube}^{\square}_{M}$.

**Definition 1.4.3°1.** We call a jet cube **conventional** if each of its dimensions has a degree equal to or lower than the previous one. We write $\mathsf{JetCubeConv}^{\boxtimes}_{M}(\omega, \vec{a})$ for the full subcategory of $\mathsf{JetCube}^{\boxtimes}_{M}(\omega, \vec{a})$ on conventional cubes.

**Corollary 1.4.3°2.** By proposition 1.4.2°12, the inclusion $\mathsf{JetCubeConv}^{\square}_M(\omega, \vec{a}) \hookrightarrow \mathsf{JetCube}^{\square}_M(\omega, \vec{a})$ is essentially surjective (in fact split essentially surjective by the existence of sorting algorithms) and thus an equivalence of categories. $\square$

**Definition 1.4.3°3.** For $M \in \{\mathsf{IPt}_2, \mathsf{Boo}\}$, any mask $\vec{a}$ and for any two objects

$$V, W \in \mathrm{Obj}(\mathsf{JetCubeConv}^{\square}_M(\mathsf{fbe}, \vec{a})),$$

we define a proof-irrelevant predicate on morphisms $\varphi : \lfloor V \rfloor \to \lfloor W \rfloor$, denoted $\vdash \varphi : V \to W$, inductively generated by the inference rules in fig. 1.1.

We discuss these inference rules one by one; their soundness and completeness will be proven in theorems 1.4.3°4 and 1.4.3°26. Specializations of these rules for symmetric degrees are given in fig. 1.2, and some rules are grouped together in fig. 1.3.

The unique morphism to the terminal cube $()$ is a jet cube morphism (TERMINAL).

We can substitute the last variable with an endpoint. If this end point is at the last dimension's source side, then the rest of the morphism lands in the $i$-opposite of $W$ (SRC:FWD, SRC:BCK), otherwise it lands in $W$ itself (TGT:FWD, TGT:BCK).

We can apply an involution to the last variable, provided that we turn around its direction (INV:FWD, INV:BCK). Doing so means that the source-side is mapped to the source-side and the target-side is mapped to the target-side, so $W$ remains unaffected.

We can apply the (opposite) $i$-twisted prism functor to a morphism (PRISM:FWD, PRISM:BCK).

If the last dimension of our target cube is of the form $\mathbf{i} : (\!|\leftrightarrow_i|\!)$, then we know that our cube is in the image of $\mathsf{USym}^{\boxdot}_i$, and we can proceed using the adjunction $\mathsf{FSym}^{\boxdot}_i \dashv \mathsf{USym}^{\boxdot}_i$ (SYMMETRIZE). This turns our last dimension into $\mathbf{i} : (\!|\frown_i|\!)$ which is a special case of both $\mathbf{i} : (\!|\rightarrow_i|\!)$ and $\mathbf{i} : (\!|\leftarrow_i|\!)$, so we can proceed by using the FWD and BCK rules of the calculus.

We can weaken w.r.t. the last dimension (WKN) of the source cube, but some caution is required. At the source-side of the last dimension, we find $\mathsf{Op}^{\boxdot}_i(V)$, whereas at the target-side we have $V$. Thus, $\varphi$ needs to be a morphism of jet cubes from $\mathsf{Op}^{\boxdot}_i(V) \to W$ as well as from $V \to W$. This can be achieved by asking that $\varphi$ starts from $\mathsf{SymCl}^{\boxdot}_i(V)$, which can be thought of as a join of $\mathsf{Op}^{\boxdot}_i(V)$ and $V$. In the case of an equijet dimension, $\mathsf{SymCl}^{\boxdot}_i(V) = V = \mathsf{Op}^{\boxdot}_i(V)$.

We can exchange variables of the same symmetric degree $i$ (EXCHANGE). Note that conventionality implies that all variables in $U_1$ are also of type $(\!|\leftrightarrow_i|\!)$.

We can substitute the last variable with a variable of a weaker (higher) degree in either direction (or of equijet dimension). Inspired by proposition 1.2.1°5, we choose to use terminology from pro-arrow equipments and refer to this action as creating a companion when the direction of the arrow remains the same ($P = Q$), and a conjoint when it reverses ($\{P, Q\} = \{\rightarrow, \leftarrow\}$); we introduce the term **concursor** (CONCURSOR) as the common generalization of companions, conjoints, and their symmetric counterpart **equiconcursors** ($P = \leftrightarrow$). Some measures of caution need to be taken however, which we consider in the case of forward companions ($P = Q = \rightarrow$), where we wish to derive $(\varphi, \mathbf{j}/\mathbf{i}) : (U, \mathbf{j} : (\!|\rightarrow_j|\!), V) \to (W, \mathbf{i} : (\!|\rightarrow_i|\!))$. First of all, we need to enforce affineness and make sure that $\varphi$ does not use the variable $\mathbf{j}$, so we will have $\varphi : (\lfloor U \rfloor, \lfloor V \rfloor) \to \lfloor W \rfloor$. Now let us look at what happens when we set $\mathbf{i}$ and $\mathbf{j}$ to 0 or to 1:

$$\begin{tikzcd}
(\mathsf{Op}^{\boxdot}_j(U), V) \arrow[r, dotted, "\varphi"] \arrow[d, "(0/\mathbf{j})"'] & \mathsf{Op}^{\boxdot}_i(W) \arrow[d, "(0/\mathbf{i})"] \\
(U, \mathbf{j} : (\!|\rightarrow_j|\!), V) \arrow[r, "(\varphi, \mathbf{j}/\mathbf{i})"'] & (W, \mathbf{i} : (\!|\rightarrow_i|\!)) \\
(U, V) \arrow[r, dotted, "\varphi"'] \arrow[u, "(1/\mathbf{j})"] & W \arrow[u, "(1/\mathbf{i})"']
\end{tikzcd}$$

So $\varphi$ needs to be both a morphism of jet cubes from $(U, V)$ to $W$ and from $(\mathsf{Op}^{\boxdot}_j(U), V)$ to $\mathsf{Op}^{\boxdot}_i(W)$ or equivalently from $\mathsf{Op}^{\boxdot}_i(\mathsf{Op}^{\boxdot}_j(U), V)$ to $W$. This can be achieved by asking that $\varphi$ starts from $\mathsf{SymCl}_i(\mathsf{SymCl}_j(U), V)$, which can be thought of as a join of $(U, V)$ and $\mathsf{Op}^{\boxdot}_i(\mathsf{Op}^{\boxdot}_j(U), V)$.

$$\textsc{Terminal}\quad \frac{}{\vdash () : V \to ()}$$

$$\textsc{Src:Fwd}\quad \frac{\vdash \varphi : V \to \mathsf{Op}_i^{\boxdot}(W)}{\vdash (\varphi, 0/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)} \qquad \textsc{Src:Bck}\quad \frac{\vdash \varphi : V \to \mathsf{Op}_i^{\boxdot}(W)}{\vdash (\varphi, 1/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis)}$$

$$\textsc{Tgt:Fwd}\quad \frac{\vdash \varphi : V \to W}{\vdash (\varphi, 1/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)} \qquad \textsc{Tgt:Bck}\quad \frac{\vdash \varphi : V \to W}{\vdash (\varphi, 0/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis)}$$

$$\textsc{Inv:Fwd}\quad \frac{\vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis)}{\vdash (\varphi, \neg t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)} \qquad \textsc{Inv:Bck}\quad \frac{\vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)}{\vdash (\varphi, \neg t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis)}$$

$$\textsc{Prism:Fwd}\quad \frac{\vdash \varphi : V \to W}{\vdash (\varphi, \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)} \qquad \textsc{Prism:Bck}\quad \frac{\vdash \varphi : V \to W}{\vdash (\varphi, \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis)}$$

$$\textsc{Symmetrize}\quad \frac{\vdash \varphi : \mathsf{FSym}_i^{\boxdot} V \to W}{\vdash \varphi : V \to \mathsf{USym}_i^{\boxdot} W}$$

$$\textsc{Wkn}\quad \frac{\vdash \varphi : \mathsf{SymCl}_i^{\boxdot}(V) \to W \qquad R \in \{\rightarrowtriangle, \leftarrowtriangle, \leftrightarrowtriangle\}}{\vdash (\varphi, \mathbf{i}/\oslash) : (V, \mathbf{i} : \llparenthesis R_i \rrparenthesis) \to W}$$

$$\textsc{Exchange}\quad \frac{\vdash \varphi : (V, \mathbf{j} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis, U_1, \mathbf{i} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis, U_2) \to W}{\vdash \varphi : (V, \mathbf{i} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis, U_1, \mathbf{j} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis, U_2) \to W}$$

$$\textsc{Concursor}\quad \frac{P \in \{\rightarrowtriangle, \leftarrowtriangle, \leftrightarrowtriangle\} \qquad Q \in \{\rightarrowtriangle, \leftarrowtriangle\} \qquad j > i \qquad \vdash \varphi : \mathsf{SymCl}_i^{\boxdot}(\mathsf{SymCl}_j^{\boxdot} U, V) \to W}{\vdash (\varphi, \mathbf{j}/\mathbf{i}) : (U, \mathbf{j} : \llparenthesis P_j \rrparenthesis, V) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}$$

$$\textsc{Conn:Prism:Src-Neutral}\quad \frac{(Q, \diamondsuit) \in \{(\rightarrowtriangle, \vee), (\leftarrowtriangle, \wedge)\} \qquad \vdash \varphi : \mathsf{SymCl}_i^{\boxdot} V \to W \qquad \vdash (\varphi, t/\mathbf{i}) : \mathsf{Op}_i^{\boxdot} V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}{\vdash (\varphi, t \diamondsuit \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\ \mathsf{Boo}$$

$$\textsc{Conn:Prism:Tgt-Neutral}\quad \frac{(Q, \diamondsuit) \in \{(\rightarrowtriangle, \wedge), (\leftarrowtriangle, \vee)\} \qquad \vdash \varphi : \mathsf{SymCl}_i^{\boxdot} V \to W \qquad \vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}{\vdash (\varphi, t \diamondsuit \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\ \mathsf{Boo}$$

$$\textsc{Conn:Prism-Inv:Src-Neutral}\quad \frac{(Q, \diamondsuit, P) \in \{(\rightarrowtriangle, \vee, \leftarrowtriangle), (\leftarrowtriangle, \wedge, \rightarrowtriangle)\} \qquad \vdash \varphi : \mathsf{SymCl}_i^{\boxdot} V \to W \qquad \vdash (\varphi, t/\mathbf{i}) : \mathsf{Op}_i^{\boxdot} V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}{\vdash (\varphi, t \diamondsuit \neg\mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis P_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\ \mathsf{Boo}$$

$$\textsc{Conn:Prism-Inv:Tgt-Neutral}\quad \frac{(Q, \diamondsuit, P) \in \{(\rightarrowtriangle, \wedge, \leftarrowtriangle), (\leftarrowtriangle, \vee, \rightarrowtriangle)\} \qquad \vdash \varphi : \mathsf{SymCl}_i^{\boxdot} V \to W \qquad \vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}{\vdash (\varphi, t \diamondsuit \neg\mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis P_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\ \mathsf{Boo}$$

$$\textsc{Conn:Degree-Symmetric}\quad \frac{Q \in \{\rightarrowtriangle, \leftarrowtriangle\} \qquad \diamondsuit \in \{\vee, \wedge\} \qquad \vdash (\varphi, s/\mathbf{i}) : \mathsf{SymCl}_i^{\boxdot} V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \qquad \vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}{\vdash (\varphi, t \diamondsuit s/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\ \mathsf{Boo}$$

Figure 1.1: A calculus of affine fbe-jet-cube morphisms, for the monads $\mathsf{IPt}_2$ and Boo. See fig. 1.2 for specializations of these rules to symmetric degrees and fig. 1.3 for unified versions of the specialized forward/backward rules.

$$\textsc{endpoint:sym}\ \frac{\vdash \varphi : V \to W \qquad c \in \{0,1\}}{\vdash (\varphi, c/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis)} \qquad \textsc{inv:sym}\ \frac{\vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis)}{\vdash (\varphi, \neg t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis)}$$

$$\textsc{prism:sym}\ \frac{\vdash \varphi : V \to W}{\vdash (\varphi, \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis)} \qquad \textsc{wkn:sym}\ \frac{\vdash \varphi : V \to W}{\vdash (\varphi, \mathbf{i}/\oslash) : (V, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis) \to W}$$

$$\textsc{exchange:sym}\ \frac{\vdash \varphi : (V, \mathbf{j} : \llparenthesis \frown_i \rrparenthesis, U_1, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis, U_2) \to W}{\vdash \varphi : (V, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis, U_1, \mathbf{j} : \llparenthesis \frown_i \rrparenthesis, U_2) \to W} \qquad \textsc{conn:sym}\ \frac{\begin{array}{l}\Diamond \in \{\vee, \wedge\} \\ \vdash (\varphi, s/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis) \\ \vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis)\end{array}}{\vdash (\varphi, t \Diamond s/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis)}\,\mathsf{Boo}$$

Figure 1.2: Symmetric specializations of the rules in fig. 1.1. Note that the rules CONN:PRISM:$*$ become a special case of CONN:DEGREE-SYMMETRIC. We omit TERMINAL which specifies no degree, CONCURSOR which specifies two, and SYMMETRIZE which already places constraints on the anpolarity $a_i$.

$$\textsc{src}\ \frac{\begin{array}{l}(Q, c) \in \{(\multimap, 0), (\multimapinv, 1)\} \\ \vdash \varphi : V \to \mathsf{Op}_i^{\boxdot}(W)\end{array}}{\vdash (\varphi, c/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)} \qquad \textsc{tgt}\ \frac{\begin{array}{l}(Q, c) \in \{(\multimap, 1), (\multimapinv, 0)\} \\ \vdash \varphi : V \to W\end{array}}{\vdash (\varphi, c/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}$$

$$\textsc{inv}\ \frac{\begin{array}{l}\{P, Q\} = \{\multimap, \multimapinv\} \\ \vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis P_i \rrparenthesis)\end{array}}{\vdash (\varphi, \neg t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)} \qquad \textsc{prism}\ \frac{\vdash \varphi : V \to W \qquad Q \in \{\multimap, \multimapinv\}}{\vdash (\varphi, \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}$$

Figure 1.3: Unified versions of the specialized forward/backward rules in fig. 1.1.

The last five rules involve connections (conjunction and disjunction) and only apply if $M = \mathsf{Boo}$, as $\mathsf{IPt}_2$ does not provide these operations. Due to the twisted nature of the twisted prism functor, it turns out that we can substitute the last variable of the target cube only with a connection of which one operand (say the right one) is either (the negation of) the last variable of the source cube, or an expression depending only on $i$-symmetric variables.[7]

In either case, it turns out that whether the last term reduces to 0 or 1 on any end point of the source cube, is sufficiently irregular that $i$-directed terms in the remainder of $\varphi$ can only depend on $i$-symmetric variables. The promotion of all variables of degree $i$ to equijet variables in the context of $\varphi$ in at least one of the premises, precludes their usage in $i$-directed terms.

In the latter case, we can use CONN:DEGREE-SYMMETRIC, where $s$ is also checked in the symmetrized context.

In the former case, we get to apply a rule that combines a connection, possibly an inversion, and the PRISM rules. The main point worth remarking upon is that the behaviour of $t$ only matters when the other operand reduces to the neutral element of the connection at hand. Depending on this, we decide whether $t$ must be checked in the $i$-opposite context or not. For example, in CONN:PRISM:SRC-NEUTRAL, if $Q = \multimap$ and $\Diamond = \vee$, then the neutral element is 0, so the behaviour of $t$ only matters when the other operand is 0. This means that we are coming from the source-side of $\mathbf{i}$, i.e. from $\mathsf{Op}_i^{\boxdot} V$. This distinction leads to four different rules (CONN:PRISM:SRC-NEUTRAL, CONN:PRISM:TGT-NEUTRAL, CONN:PRISM-INV:SRC-NEUTRAL, CONN:PRISM-INV:SRC-NEUTRAL).

[7] This is formalized in lemma 1.4.3°23.

It is worth pointing out that if $a_i = \bigcirc$, then CONN:DEGREE-SYMMETRIC specializes to the rule CONN:SYM (fig. 1.2) that is sufficiently general to also subsume the symmetric specializations of the other connection rules.

### 1.4.3 (a) Soundness

**Theorem 1.4.3°4** (Soundness)**.** If a morphism $\varphi : \lfloor V \rfloor \to \lfloor W \rfloor$ satisfies the predicate $\vdash \varphi : V \to W$ from definition 1.4.3°3, then it actually arises as the image $\varphi = \lfloor \hat{\varphi} \rfloor$ of a morphism $\hat{\varphi} : V \to W$.

**Remark 1.4.3°5.** The derivation rules in our calculus are all natural w.r.t. $V$ and $W$. Indeed, since $\lfloor \sqcup \rfloor$ is faithful, this is simply inherited from the underlying operations on cubes.

*Proof.* Note that what really needs to be proven is that $\vdash \varphi : V \to W$ implies that $\mathsf{EP}(\varphi)$, which a priori is a function from the set $\mathsf{EP}(\lfloor V \rfloor) = U(\mathsf{JEP}(V))$ to $\mathsf{EP}(\lfloor W \rfloor) = U(\mathsf{JEP}(W))$, is in fact a morphism of jet sets $\mathsf{JEP}(V) \to \mathsf{JEP}(W)$. We prove this, of course, by induction on the derivation of the inductive predicate.

- For TERMINAL, note that $\mathsf{JEP}(())$ is the terminal jet set.
- For SRC:FWD, SRC:BCK, TGT:FWD and TGT:BCK, this follows immediately from definition 1.2.2°2.
- For INV:FWD, by postcomposition, it suffices to show that $\zeta : (\mathrm{id}_W, \neg\mathbf{i}/\mathbf{i}) : \lfloor (W, i : \llparenthesis \leftarrowtriangle_i \rrparenthesis) \rfloor \to \lfloor (W, i : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \rfloor$ is a morphism of jet cubes, i.e. that $\mathsf{EP}(\zeta) : (\vec{w}, u) \mapsto (\vec{w}, \neg u)$ is a morphism of jet sets $\mathsf{JEP}(W) \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis \to \mathsf{JEP}(W) \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis$.
  Let $(\vec{w}, u) \rightarrowtriangle_j (\vec{w}', u')$ in $\mathsf{JEP}(W) \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis$. Then by definition 1.2.2°2 of the opposite $i$-twisted prism, there are 3 possibilities:
  - We have $u = u' = 0$ and $\vec{w} \rightarrowtriangle_j \vec{w}'$ in $\mathsf{JEP}(W)$. In that case, we also have the required jet between the images $(\vec{w}, 1) \rightarrowtriangle_j (\vec{w}', 1)$ in $\mathsf{JEP}(W) \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis$.
  - We have $u = u' = 1$ and $\vec{w} \rightarrowtriangle_j \vec{w}'$ in $\mathsf{Op}_i(\mathsf{JEP}(W))$. In that case, we also have the required jet between the images $(\vec{w}, 0) \rightarrowtriangle_j (\vec{w}', 0)$ in $\mathsf{JEP}(W) \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis$.
  - We have $j = i$, $u = 1$, $u' = 0$ and $\vec{w} = \vec{w}'$. In that case, we also have the required jet between the images $(\vec{w}, 0) \rightarrowtriangle_i (\vec{w}, 1)$ in $\mathsf{JEP}(W) \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis$.

  The proof of soundness of INV:BCK is analogous.
- Soundness of PRISM:FWD and PRISM:BCK was already established by proposition 1.4.2°11.
- Soundness of SYMMETRIZE follows from the adjunction established in proposition 1.4.2°11.
- We prove soundness of WKN by precomposition with a jet cube morphism that erases to $(\mathrm{id}, \mathbf{i}/\oslash) : \lfloor (V, \mathbf{i} : \llparenthesis R_i \rrparenthesis) \rfloor \to \mathsf{SymCl}_i \lfloor V \rfloor$. Thus, we need to prove that $\mathsf{EP}(\mathrm{id}, \mathbf{i}/\oslash) : (\vec{v}, w) \mapsto \vec{v}$ is a jet set morphism $\mathsf{JEP}(V, \mathbf{i} : \llparenthesis R_i \rrparenthesis) \to \mathsf{JEP}(\mathsf{SymCl}_i^{\boxdot}(V))$. Let $(\vec{v}, w) \rightarrowtriangle_j (\vec{v}', w')$ in $\mathsf{JEP}(V, \mathbf{i} : \llparenthesis R_i \rrparenthesis)$. Then there are two possibilities:
  - $w = w'$ and $(\vec{v}, w) \rightarrowtriangle_j (\vec{v}', w)$. The latter implies $\vec{v} \rightarrowtriangle_j \vec{v}'$ in $\mathsf{JEP}(V)$ if $j \neq i$ and $\vec{v} \rightleftarrows_i \vec{v}'$ if $j = i$. Moving to $\mathsf{JEP}(\mathsf{SymCl}_i^{\boxdot}(V))$, we get $\vec{v} \rightarrowtriangle_j \vec{v}'$ in all cases, as required.
  - $j = i$, $\vec{v} = \vec{v}'$ and $w \rightarrowtriangle_i w'$. In this case we have $\vec{v} \rightarrowtriangle_j \vec{v}$ in $\mathsf{JEP}(\mathsf{SymCl}_i^{\boxdot}(V))$ by reflexivity.
- Recalling definition 1.4.2°5, soundness of EXCHANGE follows from proposition 1.4.2°13. (Note that, since we are dealing with conventional cubes, all variables in $U_1$ also have type $\llparenthesis \leftrightarrowtriangle_i \rrparenthesis$.)
- We prove soundness of CONCURSOR. Assume that $\varphi$ is a jet cube morphism $\mathsf{SymCl}_i^{\boxdot}(\mathsf{SymCl}_j^{\boxdot} U, V) \to W$ and $j > i$. We prove that $(\varphi, \mathbf{j}/\mathbf{i})$ is a jet cube morphism $(U, \mathbf{j} : \llparenthesis P_j \rrparenthesis, V) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)$. Write $f = \mathsf{EP}(\varphi)$ and $g = \mathsf{EP}(\varphi, \mathbf{j}/\mathbf{i})$.
  Pick a jet $(\vec{u}, t, \vec{v}) \rightarrowtriangle_k (\vec{u}', t', \vec{v}')$ in $\mathsf{JEP}(U, \mathbf{j} : \llparenthesis P_j \rrparenthesis, V)$. We can assume that this jet is not reflexive. Let $\mathbf{k}$ be the variable where both hands differ. There are three possibilities:
  - If $\mathbf{k} \in U$, then we have $t = t'$, $\vec{v} = \vec{v}'$ and $(\vec{u}, t, \vec{v}) \rightarrowtriangle_k (\vec{u}', t, \vec{v})$.
    - If $k \neq j$ and $k \neq i$, this implies $(\vec{u}, \vec{v}) \rightarrowtriangle_k (\vec{u}', \vec{v})$ in $\mathsf{JEP}(U, V)$ and therefore also in $\mathsf{JEP}(\mathsf{SymCl}_i^{\boxdot}(\mathsf{SymCl}_j^{\boxdot} U, V))$, whence $f(\vec{u}, \vec{v}) \rightarrowtriangle_k f(\vec{u}', \vec{v})$, whence $g(\vec{u}, t, \vec{v}) = (f(\vec{u}, \vec{v}), t) \rightarrowtriangle_k (f(\vec{u}', \vec{v}), t) = g(\vec{u}', t, \vec{v})$.
    - If $k = i$ or $k = j$, this implies $(\vec{u}, \vec{v}) \rightleftarrows_k (\vec{u}', \vec{v})$ in $\mathsf{JEP}(U, V)$ and therefore $(\vec{u}, \vec{v}) \leftrightarrowtriangle_k (\vec{u}', \vec{v})$ in $\mathsf{JEP}(\mathsf{SymCl}_i^{\boxdot}(\mathsf{SymCl}_j^{\boxdot} U, V))$, whence $f(\vec{u}, \vec{v}) \leftrightarrowtriangle_k f(\vec{u}', \vec{v})$, whence $g(\vec{u}, t, \vec{v}) = (f(\vec{u}, \vec{v}), t) \leftrightarrowtriangle_k (f(\vec{u}', \vec{v}), t) = g(\vec{u}', t, \vec{v})$.

  - If $\mathbf{k} = \mathbf{j}$, then we have $k = j$, $\vec{u} = \vec{u}'$, $t \neq t'$ and $\vec{v} = \vec{v}'$. Then $g(\vec{u}, t, \vec{v}) = (f(\vec{u}, \vec{v}), t) \rightleftarrows_i (f(\vec{u}, \vec{v}), t') = g(\vec{u}, t', \vec{v})$ which implies $g(\vec{u}, t, \vec{v}) \rightarrowtriangle_j g(\vec{u}, t', \vec{v})$ since $j > i$.
  - If $\mathbf{k} \in V$, then we have $t = t'$, $\vec{u} = \vec{u}'$ and $\vec{v} \rightarrowtriangle_k \vec{v}'$ in $\mathsf{JEP}(V)$.
    - If $k \neq i$, this implies $(\vec{u}, \vec{v}) \rightarrowtriangle_k (\vec{u}, \vec{v}')$ in $\mathsf{JEP}(\mathsf{SymCl}_i^{\boxdot}(\mathsf{SymCl}_j^{\boxdot} U, V))$, whence $f(\vec{u}, \vec{v}) \rightarrowtriangle_k f(\vec{u}, \vec{v}')$, whence $g(\vec{u}, t, \vec{v}) = (f(\vec{u}, \vec{v}), t) \rightarrowtriangle_k (f(\vec{u}, \vec{v}'), t) = g(\vec{u}, t, \vec{v}')$.
    - If $k = i$, this implies $(\vec{u}, \vec{v}) \leftrightarrowtriangle_i (\vec{u}, \vec{v}')$ in $\mathsf{JEP}(\mathsf{SymCl}_i^{\boxdot}(\mathsf{SymCl}_j^{\boxdot} U, V))$, whence $f(\vec{u}, \vec{v}) \leftrightarrowtriangle_i f(\vec{u}, \vec{v}')$, whence $g(\vec{u}, t, \vec{v}) = (f(\vec{u}, \vec{v}), t) \leftrightarrowtriangle_i (f(\vec{u}, \vec{v}'), t) = g(\vec{u}, t, \vec{v}')$.

- We prove soundness of CONN:PRISM:SRC-NEUTRAL for the case where $Q = \rightarrowtriangle$ and $\Diamond = \vee$, the other case is proven analogously. Assume that $\varphi$ is a jet cube morphism $\mathsf{SymCl}_i^{\boxdot} V \to W$ and $(\varphi, t/\mathbf{i})$ is a jet cube morphism $\mathsf{Op}_i^{\boxdot} V \to (W, \mathbf{i} : (\!|{\mapstochar\rightarrowtriangle_i}|\!))$. We prove that $(\varphi, t \vee \mathbf{i}/\mathbf{i})$ is a jet cube morphism $(V, \mathbf{i} : (\!|{\mapstochar\rightarrowtriangle_i}|\!)) \to (W, \mathbf{i} : (\!|{\mapstochar\rightarrowtriangle_i}|\!))$. Write $f = \mathsf{EP}(\varphi)$ and $g = \mathsf{EP}(\varphi, t \vee \mathbf{i}/\mathbf{i})$ and $h = \mathsf{EP}(\varphi, t/\mathbf{i})$.
  Pick a non-reflexive jet $(\vec{v}, u) \rightarrowtriangle_j (\vec{v}', u')$ in $\mathsf{JEP}(V, \mathbf{i} : (\!|{\mapstochar\rightarrowtriangle_i}|\!))$. Let $\mathbf{k}$ be the variable where both hands defer. There are two possibilities:
  - If $\mathbf{k} = \mathbf{i}$, then $j = i$, $\vec{v} = \vec{v}'$, $u = 0$ and $u' = 1$. In this case, $(t \vee \mathbf{i})\langle \vec{v}, 1 \rangle = 1$, so that there is necessarily an $i$-jet $g(\vec{v}, 0) = (f(\vec{v}), (t \vee \mathbf{i})\langle \vec{v}, 0 \rangle) \rightarrowtriangle_i (f(\vec{v}), 1) = g(\vec{v}, 1)$.
  - If $\mathbf{k} \in V$, then $u = u'$ and $(\vec{v}, u) \rightarrowtriangle_j (\vec{v}', u)$. Let $\mathbf{l}$ be the variable in $(W, \mathbf{i} : (\!|{\mapstochar\rightarrowtriangle_i}|\!))$ such that $\mathbf{l}\langle \varphi, t \vee \mathbf{i} \rangle$ depends on $\mathbf{k}$. If there is no such variable, then we are done.
    - If $\mathbf{l} = \mathbf{i}$, then $\mathbf{k}$ occurs in $t$.
      - If $u = 1$, then $g(\vec{v}, 1) = (f(\vec{v}), 1) \overset{=}{\rightarrowtriangle}_j (f(\vec{v}'), 1) = g(\vec{v}', 1)$ as required.
      - If $u = 0$, then we have $\vec{v} \rightarrowtriangle_j \vec{v}'$ in $\mathsf{JEP}(\mathsf{Op}_i^{\boxdot} V)$. Because $(\varphi, t/\mathbf{i})$ is a jet cube morphism $\mathsf{Op}_i^{\boxdot} V \to (W, \mathbf{i} : (\!|{\mapstochar\rightarrowtriangle_i}|\!))$, we get $g(\vec{v}, 0) = h(\vec{v}) \rightarrowtriangle_j h(\vec{v}') = g(\vec{v}', 0)$ as required.
    - If $\mathbf{l} \in W$, then $\mathbf{k}$ occurs in $\varphi$. Define $z = (t \vee \mathbf{i})\langle \vec{v}, u \rangle = (t \vee \mathbf{i})\langle \vec{v}', u \rangle$. We have $g(\vec{v}, u) = (f(\vec{v}), z)$ and $g(\vec{v}', u) = (f(\vec{v}'), z)$.
      - If $j = i$, then we have $\vec{v} \leftrightarrowtriangle_i \vec{v}'$ in $\mathsf{JEP}(\mathsf{SymCl}_i^{\boxdot} V)$, whence $f(\vec{v}) \leftrightarrowtriangle_i f(\vec{v}')$ in $\mathsf{JEP}(W)$, whence $g(\vec{v}, u) = (f(\vec{v}), z) \rightarrowtriangle_i (f(\vec{v}'), z) = g(\vec{v}', u)$ in $\mathsf{JEP}(W, \mathbf{i} : (\!|{\mapstochar\rightarrowtriangle_i}|\!))$.
      - If $j \neq i$, then we have $\vec{v} \rightarrowtriangle_j \vec{v}'$ in $\mathsf{JEP}(\mathsf{SymCl}_i^{\boxdot} V)$, whence $f(\vec{v}) \rightarrowtriangle_j f(\vec{v}')$ in $\mathsf{JEP}(W)$, whence $g(\vec{v}, u) = (f(\vec{v}), z) \rightarrowtriangle_j (f(\vec{v}'), z) = g(\vec{v}', u)$ in $\mathsf{JEP}(W, \mathbf{i} : (\!|{\mapstochar\rightarrowtriangle_i}|\!))$.
- Soundness of CONN:PRISM:TGT-NEUTRAL is proven analogously to that of CONN:PRISM:SRC-NEUTRAL.
- Soundness of CONN:PRISM-INV:SRC-NEUTRAL is proven from soundness of CONN:PRISM:SRC-NEUTRAL by precomposing the result with $(\mathrm{id}_V, \neg\mathbf{i}/\mathbf{i})$ which is a jet cube morphism $(V, \mathbf{i} : (\!|P_i|\!)) \to (V, \mathbf{i} : (\!|Q_i|\!))$.
- Soundness of CONN:PRISM-INV:TGT-NEUTRAL is similarly proven from soundness of CONN:PRISM:TGT-NEUTRAL.
- We prove soundness of CONN:DEGREE-SYMMETRIC. Assume that
  - $(\varphi, s/\mathbf{i})$ is a jet cube morphism $\mathsf{SymCl}_i^{\boxdot} V \to (W, \mathbf{i} : (\!|Q_i|\!))$,
  - $(\varphi, t/\mathbf{i})$ is a jet cube morphism $V \to (W, \mathbf{i} : (\!|Q_i|\!))$.

  We prove that $(\varphi, s \Diamond t/\mathbf{i})$ is a jet cube morphism $V \to (W, \mathbf{i} : (\!|Q_i|\!))$. Write

  $$f = \mathsf{EP}(\varphi), \qquad g = \mathsf{EP}(\varphi, s/\mathbf{i}), \qquad h = \mathsf{EP}(\varphi, t/\mathbf{i}), \qquad d = \mathsf{EP}(\varphi, s \Diamond t/\mathbf{i}).$$

  Pick a non-reflexive jet $\vec{v} \rightarrowtriangle_j \vec{v}'$ in $\mathsf{JEP}(V)$; we prove that $d(\vec{v}) \rightarrowtriangle d(\vec{v}')$ in $\mathsf{JEP}(W, \mathbf{i} : (\!|Q_i|\!))$. Let $\mathbf{k}$ be the variable of $V$ where $\vec{v}$ and $\vec{v}'$ differ. There are four possible cases:
  - If $\varphi$, $s$ and $t$ do not depend on $\mathbf{k}$ then the target jet is reflexive.
  - If $\varphi$ or $s$ depends on $\mathbf{k}$, then we have $t\langle \vec{v} \rangle = t\langle \vec{v}' \rangle =: t_0$.
    - If $j \neq i$, then we have $\vec{v} \rightarrowtriangle_j \vec{v}'$ in $\mathsf{JEP}(\mathsf{SymCl}_i^{\boxdot} V)$, whence $(f(\vec{v}), s\langle \vec{v} \rangle) = g(\vec{v}) \rightarrowtriangle_j g(\vec{v}') = (f(\vec{v}'), s\langle \vec{v}' \rangle)$ in $\mathsf{JEP}(W, \mathbf{i} : (\!|Q_i|\!))$. Taking a connection with $t_0$ does not influence the direction of the arrows to the left of $\mathbf{i}$, nor of the arrows at $\mathbf{i}$. Hence we get $d(\vec{v}) = (f(\vec{v}), s\langle \vec{v} \rangle \Diamond t_0) \rightarrowtriangle_j (f(\vec{v}'), s\langle \vec{v}' \rangle \Diamond t_0) = d(\vec{v}')$.
    - If $j = i$, then we have $\vec{v} \leftrightarrowtriangle_i \vec{v}'$ in $\mathsf{JEP}(\mathsf{SymCl}_i^{\boxdot} V)$, whence $(f(\vec{v}), s\langle \vec{v} \rangle) = g(\vec{v}) \leftrightarrowtriangle_i g(\vec{v}') = (f(\vec{v}'), s\langle \vec{v}' \rangle)$ in $\mathsf{JEP}(W, \mathbf{i} : (\!|Q_i|\!))$. Hence we get $d(\vec{v}) = (f(\vec{v}), s\langle \vec{v} \rangle \Diamond t_0) \leftrightarrowtriangle_i (f(\vec{v}'), s\langle \vec{v}' \rangle \Diamond t_0) = d(\vec{v}')$.

– If $t$ depends on $\mathbf{k}$, then we have $f(\vec{v}) = f(\vec{v}') =: f_0$ and $s\langle\vec{v}\rangle = s\langle\vec{v}'\rangle =: s_0$. We get $(f_0, t\langle\vec{v}\rangle) = h(\vec{v}) \twoheadrightarrow_j h(\vec{v}') = (f_0, t\langle\vec{v}'\rangle)$. Taking a connection with $s_0$ yields $d(\vec{v}) = (f_0, s_0 \diamondsuit t\langle\vec{v}\rangle) \twoheadrightarrow_j (f_0, s_0 \diamondsuit t\langle\vec{v}'\rangle) = d(\vec{v}')$. □

#### 1.4.3 (b) Lemmas for Completeness

Proving completeness for the $\mathsf{IPt}_2$ monad is fairly straightforward, but for the cases involving connections (conjunctions and disjunctions), we need a couple of helper lemmas.

**Boolean reduction** In this section, we establish a normal form for affine boolean terms.

**Definition 1.4.3°6.** Boolean **terms** $t \in \mathsf{Boo}(X)$ are equivalence classes $t = [e]$ of boolean **expressions** $e \in \mathsf{BooE}(X)$, which are defined as abstract syntax trees made up of 0, 1, $\vee$, $\wedge$, $\neg$ and elements of $X$. Thanks to commutativity and associativity, we regard $\vee$ and $\wedge$ as having a *multiset* of operands with at least two elements. The other operations have the usual arities.

We define a reduction algorithm that reduces an expression $e$ to $e'$ such that $[e] = [e']$:

- push all negations down to the leaves of the syntax tree,
- eliminate negations of constants,
- eliminate double negations,
- eliminate conjunctions/disjunctions with constants,
- remove parentheses of nested conjunctions / nested disjunctions.

We call an expression **normal** if it is its own reduction, i.e. if it is either a constant or a tree whose nodes are alternatingly (as we climb the tree) labeled with $\vee$ and $\wedge$ (the root can have either) and whose leaves are **literals**, where a literal is either a variable or its negation.

**Definition 1.4.3°7.** Let $d$ and $e$ be normal expressions. We say that $d$ is a **pruning** of $e$ if any of the following conditions hold (inductively):

- $d$ is a literal occurring in $e$,
- $e$ has a subexpression $e'$ which has the same root label $\diamondsuit \in \{\vee, \wedge\}$ as $d$ and such that there exists a partition of the multiset of operands of $d$, i.e.

$$d = D_1 \diamondsuit \ldots \diamondsuit D_n \quad \text{where} \quad D_i = d_{i1} \diamondsuit \ldots \diamondsuit d_{ik_i}$$

  such that every $D_i$ is a pruning of a root operand in $e'$, and moreover a single operand of $e'$ cannot be used more often than its multiplicity in the multiset of root operands of $e'$.

**Proposition 1.4.3°8.** Let $e$ be a normal expression with variables in $X$ and let $\sigma$ be a bit-assignment $\sigma : Y \to \{0, 1\}$ of the variables in $Y \subseteq X$. Let $e[\sigma]$ reduce to $d$. Then $d$ is either a constant or a pruning of $e$.

*Proof.* By induction on $e$. If $e$ is a constant or a literal, this is immediate. If $e$ is a conjunction or disjunction, then this follows from the induction hypothesis for the immediate children of $e$. □

**Lemma 1.4.3°9.** For every $t \in \mathsf{Boo}^{\#}(X)$ and $c \in \{0, 1\}$, there exists a bit assignment $\sigma : X \to \{0, 1\}$ such that $t[\sigma] = c$.

*Proof.* Pick an affine representant $e \in [t]$ and assume it is normal (reduce if it is not). Then we can prove this by induction on the height of $e$. □

**Lemma 1.4.3°10.** For every normal affine expression $e \in \mathsf{BooE}^{\#}(X)$ mentioning all and only the variables in $Z \subseteq X$, and for every $Y \subseteq X$, there exists a bit assignment $\sigma : X \setminus Y \to \{0, 1\}$ such that $e[\sigma]$ reduces to a (necessarily normal affine) expression $d$ which mentions all and only the variables in $Y \cap Z$.

*Proof.* By induction on $e$. If $e$ is a leaf (i.e. a constant or a literal), then this is trivial. If $e$ is a node with label $\diamondsuit \in \{\vee, \wedge\}$, then we invoke the induction hypothesis for all immediate subtrees mentioning variables in $Y$, and find assignments for the variables mentioned in those immediate subtrees. For immediate subtrees not mentioning variables in $Y$, we use lemma 1.4.3°9 to find an assigment that reduces this subtree to the neutral element $\iota_\diamondsuit$ of $\diamondsuit$. Combining all assignments yields the required result. ☐

**Corollary 1.4.3°11.** For every normal affine expression $e \in \mathsf{BooE}^\#(X)$ that has a leaf $\tilde{y} \in \{y, \neg y\}$ where $y \in X$, there exists a bit assignment $\sigma : X \setminus \{y\} \to \{0, 1\}$ such that $t[\sigma] = \tilde{x}$. ☐

**Corollary 1.4.3°12.** If $e$ is a normal affine expression $e \in \mathsf{BooE}^\#(X)$ depending on $y$, then every expression representing $[e]$ depends on $y$. We say that **an affine term** $t \in \mathsf{Boo}^\#(X)$ **depends on** $y \in X$ if the following equivalent conditions hold:

- All representants of $t$ depend on $y$.
- All normal affine representants of $t$ depend on $y$.
- Some normal affine representant of $t$ depends on $y$. ☐

**Definition 1.4.3°13.** If $e$ is a normal affine expression $e \in \mathsf{BooE}^\#(X)$ depending on $x, y \in X$, then we say that $x$ and $y$ are **in $\diamondsuit$-connection** (where $\diamondsuit \in \{\vee, \wedge\}$; concretely, we call this **in disjunction/conjunction**) **in** $e$ if the closest common parent node of (the negation of) $x$ and (the negation of) $y$ is labelled with $\diamondsuit$. We also write this as $\mathsf{getConn}_e(x, y) = \diamondsuit$.

**Lemma 1.4.3°14.** Let $d, e \in \mathsf{BooE}^\#(X)$ be normal affine expressions and $d$ a pruning of $e$. Let $d$ (hence $e$) depend on $x, y \in X$. Then $x$ and $y$ are in $\diamondsuit$-connection in $d$ if and only if they are in $\diamondsuit$-connection in $e$.

*Proof.* Take the closest common ancestor of $x$ and $y$ in $d$. Proving that $d$ is a pruning of $e$ involves, at some point, that the operands of $d$ mentioning $x$ and $y$ are separated. At that point, a corresponding node in $e$ was chosen with the same label; call $e'$ the subtree rooted there. The immediate subtrees of $d'$ mentioning $x$ and $y$ are prunings of different immediate subtrees of $e'$, so that the root node of $e'$ is also the closest common ancestor of $x$ and $y$ in $e$. ☐

**Lemma 1.4.3°15.** An affine boolean expression $t \in \mathsf{Boo}^\#(X)$ mentioning exactly two variables $x$ and $y$, has only a single normal representant, which is of the form $\tilde{x} \diamondsuit \tilde{y}$ with $\tilde{x}$ and $\tilde{y}$ literals mentioning $x$ and $y$, and $\diamondsuit \in \{\vee, \wedge\}$.

*Proof.* Since every node has at least two children and there are exactly two leaves, there is exactly one node. This constrains the form of the normal representant. Next, each one of these forms produces a different truth table and the truth table is a property of $t$, so there can be only one normal representant. ☐

**Lemma 1.4.3°16.** Let $t \in \mathsf{Boo}^\#(X)$ depend on $x, y \in X$, and let $\diamondsuit \in \{\vee, \wedge\}$. The following conditions are equivalent:

- $x$ and $y$ are in $\diamondsuit$-connection in some normal affine representant of $t$,
- $x$ and $y$ are in $\diamondsuit$-connection in all normal affine representants of $t$,
- $x$ and $y$ are **in $\diamondsuit$-connection in** $t$ (definition).

We also write this as $\mathsf{getConn}_t(x, y) = \diamondsuit$.

*Proof.* Let $e$ and $e'$ be normal affine representants of $t$ and let $x$ and $y$ be in $\diamondsuit$-connection in $e$. Pick an assignment $\sigma$ such that $e[\sigma]$ reduces to $d$ which depends exactly on $\{x, y\}$. By lemma 1.4.3°15, $d$ is then the unique normal affine representant of $t[\sigma]$, so that $e'[\sigma]$ also reduces to $d$. Then $d$ is a pruning of both $e$ and $e'$, so by lemma 1.4.3°14, $x$ and $y$ are in $\diamondsuit$-connection in $e'$. ☐

**Lemma 1.4.3°17.** (Not used.) Let $e \in \mathsf{BooE}^\#(X)$ be a normal affine expression depending on $x$ and $y$ where $x \neq y$. The following conditions are equivalent:

- $\mathsf{getConn}_e(x, z) = \mathsf{getConn}_e(y, z)$ for all $z \in X \setminus \{x, y\}$,

- the literals $\tilde{x}$ and $\tilde{y}$ corresponding to $x$ and $y$ occurring in $e$ are immediate siblings.

*Proof.* It is clear that the second condition implies the first. To prove the other implication, assume that $\tilde{x}$ and $\tilde{y}$ are not immediate siblings. Let $\diamondsuit = \mathsf{getConn}_e(x, y)$. Then one of them, say $\tilde{y}$, has a closer relative $\tilde{z}$ such that $\mathsf{getConn}_e(y, z) = \heartsuit \neq \diamondsuit$. This means that $e$ has a pruning of the form $d = \tilde{x} \diamondsuit (\tilde{y} \heartsuit \tilde{z})$. But then we have

$$\mathsf{getConn}_e(x, z) = \mathsf{getConn}_d(x, z) = \diamondsuit \neq \heartsuit = \mathsf{getConn}_d(y, z) = \mathsf{getConn}_e(y, z),$$

violating the assumption. □

**Lemma 1.4.3°18.** Let $X_1, X_2, \ldots, X_n \subseteq Z$ be disjoint and $n > 1$. Let $e \in \mathsf{BooE}^{\#}(Z)$ be a normal affine expression depending on all and only on variables in $X = \bigcup_i X_i$. Let $\{\diamondsuit, \heartsuit\} \in \{\vee, \wedge\}$. The following conditions are equivalent:

- both of the following conditions hold:
  - $\mathsf{getConn}_e(x, y) = \diamondsuit$ for all $x \in X_i$, $y \in X_j$, $i \neq j$,
  - each $X_i$ is $\heartsuit$-connected meaning that no further partition of $X_i$ is possible maintaining the property above,
- $e$ is of the form $e = d_1 \diamondsuit d_2 \diamondsuit \ldots \diamondsuit d_n$ with each $d_i$ depending on all and only on variables in $X_i$, and being either a leaf or a tree with root node labelled with $\heartsuit$.

*Proof.* It is clear that the second condition implies the first. To prove the other implication, assume that $e$ is not of this form. There are two possibilities:

- $e$ is of the form $e = d_1 \diamondsuit d_2 \diamondsuit \ldots \diamondsuit d_m$ but the dependencies are not as expected. Then we get a different partition of $X$ by the upward implication which we already proved. But then we can intersect both partitions, yielding a finer one, which is in contradiction with the assumption that further partitioning was not possible.
- $e$ is of the form $e = d_1 \heartsuit d_2 \heartsuit \ldots \heartsuit d_m$. Then $X$ is $\heartsuit$-connected meaning that $n = 1$, violating the assumptions. □

**Lemma 1.4.3°19.** A non-constant normal affine boolean expression $e \in \mathsf{BooE}^{\#}(X)$ is fully determined by the set of literals it mentions and $\mathsf{getConn}_e(\sqcup, \sqcup)$.

*Proof.* Let $e, e' \in \mathsf{BooE}^{\#}(X)$ be normal affine boolean expressions mentioning the same literals and such that $\mathsf{getConn}_e(\sqcup, \sqcup) = \mathsf{getConn}_{e'}(\sqcup, \sqcup)$. We prove that $e = e'$ by induction on $e$. If $e$ is a constant or a literal, the result is immediate. Otherwise, $e$ is of the form $e = d_1 \diamondsuit d_2 \diamondsuit \ldots \diamondsuit d_n$, where $\{\diamondsuit, \heartsuit\} \in \{\vee, \wedge\}$. Then lemma 1.4.3°18 partitions $X$ in $\heartsuit$-connected components. Then by the other implication of lemma 1.4.3°18, $e'$ is of the form $e' = d'_1 \diamondsuit d'_2 \diamondsuit \ldots \diamondsuit d'_n$, where $d_i$ and $d'_i$ have the same dependencies $X_i$. Thus, they must also depend on the same set of literals. Since $d_i$ is a pruning of $e$ and $d'_i$ is a pruning of $e'$, we have $\mathsf{getConn}_{d_i}(\sqcup, \sqcup) = \mathsf{getConn}_e(\sqcup, \sqcup) = \mathsf{getConn}_{e'}(\sqcup, \sqcup) = \mathsf{getConn}_{d'_i}(\sqcup, \sqcup)$ when considered on pairs of distinct variables in $X_i$. From the induction hypothesis, we conclude that $d_i = d'_i$, for all $i$, and hence $e = e'$. □

**Theorem 1.4.3°20.** Every affine boolean term $t \in \mathsf{Boo}^{\#}(Z)$ has exactly one normal affine representant.

*Proof.* Let $e, e' \in \mathsf{BooE}^{\#}(X)$ be normal affine representants of $t$. By corollary 1.4.3°12, we know that $e$ and $e'$ depend on the same set of variables $X$. By corollary 1.4.3°11, we know moreover that they mention the same literals. By lemma 1.4.3°16, we know that $\mathsf{getConn}_e(\sqcup, \sqcup) = \mathsf{getConn}_{e'}(\sqcup, \sqcup) = \mathsf{getConn}_t(\sqcup, \sqcup)$. Thus, by lemma 1.4.3°19, we conclude that $e = e'$. □

**Understanding jet cube morphisms**

**Lemma 1.4.3°21.** In $\mathsf{JetCube}^{\boxdot}_M(\mathsf{fbe}, \vec{a})$ with $M \in \{\mathsf{IPt}_2, \mathsf{Boo}\}$, the following holds: If a cube morphism $\varphi$ is a jet cube morphism $V = (V_0, \mathbf{j} : (\!|P_j|\!), V_1) \to W = (W_0, \mathbf{i} : (\!|Q_i|\!), W_1)$ with $P, Q \in \{\rightarrowtriangle, \leftarrowtriangle, \leftrightarrowtriangle\}$ and $j > i$, and if either of the following conditions hold:

- $\mathbf{j}$ appears in $\mathbf{i}\langle\varphi\rangle$,
- $\mathbf{j}$ does not appear in $\varphi$ at all,

then $\varphi$ is also a jet cube morphism $\tilde{V} := (\mathsf{SymCl}^{\boxdot}_j(V_0), \mathbf{j} : (\!|\leftrightarrowtriangle_j|\!), V_1) \to W$.

We remark that this lemma is vacuous if $a_i = \bigcirc$ or $P = \leftrightarrowtriangle$.

In words, the lemma says: When a variable of the domain of a jet cube morphism is used at a lower degree in the codomain, or not at all, then that variable and all variables of the same degree to its left can be promoted to equijet variables.

*Proof.* Let $W'$ be the cube obtained from $W$ by simply deleting all variables of degree $i$ or lower. Then the weakening morphism $\pi : W \to W'$ is a jet cube morphism. Thanks to affineness, $\pi \circ \varphi : V \to W'$ does not depend on $\mathbf{j}$. Hence, $\pi \circ \varphi \circ (0/\mathbf{j}) = \pi \circ \varphi \circ (1/\mathbf{j}) =: \rho : (\lfloor V_0 \rfloor, \lfloor V_1 \rfloor) \to \lfloor W' \rfloor$. In the category of *jet cubes* and *cube morphisms between their erasures*, we have a commutative diagram

$$\begin{tikzcd}
(\mathsf{Op}^{\boxdot}_j V_0, V_1) \arrow[r, "(0/\mathbf{j})"] \arrow[rdd, dotted, "\rho"'] & V \arrow[d, "\varphi"] & (V_0, V_1) \arrow[l, "(1/\mathbf{j})"'] \arrow[ldd, dotted, "\rho"] \\
 & W \arrow[d, "\pi"] & \\
 & W' &
\end{tikzcd}$$

where the black arrows are known to be jet cube morphisms, and hence the dotted arrows are also jet cube morphisms as jet cube morphisms compose. Thus, $\rho$ is both a jet cube morphism $(\mathsf{Op}^{\boxdot}_j V_0, V_1) \to W'$ and $(V_0, V_1) \to W'$, hence it is a jet cube morphism $(\mathsf{SymCl}^{\boxdot}_j V_0, V_1) \to W'$.

We now show that $\varphi$ is a jet cube morphism $\tilde{V} \to W$, i.e. that $f := \mathsf{EP}(\varphi)$ is a jet set morphism $\mathsf{JEP}(\tilde{V}) \to \mathsf{JEP}(W)$. Pick a non-reflexive jet $\vec{v} = (\vec{v}_0, u, \vec{v}_1) \rightarrowtriangle_k \vec{v}' = (\vec{v}'_0, u', \vec{v}'_1)$ in $\tilde{V}$; we show that $f(\vec{v}) \rightarrowtriangle_k f(\vec{v}')$. If $k \neq j$, then $\vec{v} \rightarrowtriangle_k \vec{v}'$ is also a jet in $V$ and therefore preserved by $f$. Thus, we can assume that $k = j$. Let $\mathbf{k}$ be the unique variable where $\vec{v}$ and $\vec{v}'$ differ. There are three possibilities

$\boxed{\mathbf{k} \in V_0}$ In this case, we have $\vec{v}_0 \rightleftarrows_j \vec{v}'_0$ in $V_0$, $u = u'$, $\vec{v}_1 = \vec{v}'_1$, $\vec{v} \rightleftarrows_j \vec{v}'$ in $V$ and $f(\vec{v}) \rightleftarrows_j f(\vec{v}')$ in $W$. If $\varphi$ does not depend on $\mathbf{k}$, then $f(\vec{v}) = f(\vec{v}')$ and therefore $f(\vec{v}) \rightarrowtriangle_j f(\vec{v}')$. So we assume that $\varphi$ depends on $\mathbf{k}$; let $\mathbf{l}$ be the variable such that $\mathbf{l}\langle\varphi\rangle$ depends on $\mathbf{k}$.

- If $\mathbf{l}$ is of degree $\ell \leq i < j$, then $f(\vec{v}) \rightleftarrows_j f(\vec{v}')$ is only possible if $f(\vec{v}) \rightleftarrows_\ell f(\vec{v}')$ which implies $f(\vec{v}) \leftrightarrowtriangle_j f(\vec{v}')$.
- If $\mathbf{l}$ is of degree $\ell > i$, then we have $\mathsf{EP}(\rho)(\vec{v}_0, \vec{v}_1) \overset{\neq}{\leftrightarrowtriangle}_j \mathsf{EP}(\rho)(\vec{v}'_0, \vec{v}_1)$ because $\rho$ is a jet cube morphism $(\mathsf{SymCl}^{\boxdot}_j V_0, V_1) \to W'$. Since $\mathsf{EP}(\rho) = \mathsf{EP}(\pi) \circ \mathsf{EP}(\varphi) \circ \mathsf{EP}(c/\mathbf{j})$ for any $c \in \{0, 1\}$ and the components forgotten by $\mathsf{EP}(\pi)$ are identical, we have $f(\vec{v}) \leftrightarrowtriangle_j f(\vec{v}')$.

$\boxed{\mathbf{k} = \mathbf{j}}$ In this case, we have $\vec{v}_0 = \vec{v}'_0$, $\vec{v}_1 = \vec{v}'_1$, and $\vec{v} \rightleftarrows_j \vec{v}'$ in $V$. Therefore we get $f(\vec{v}) \rightleftarrows_j f(\vec{v}')$ and these vectors differ at their value for $\mathbf{i}$, which has degree $i$, so this is only possible if $f(\vec{v}) \rightleftarrows_i f(\vec{v}')$, which implies $f(\vec{v}) \leftrightarrowtriangle_j f(\vec{v}')$.

$\boxed{\mathbf{k} \in V_1}$ Then $\vec{v} \rightarrowtriangle_j \vec{v}'$ holds in $V$ and is therefore preserved by $f = \mathsf{EP}(\varphi)$. $\square$

**Lemma 1.4.3°22.** In $\mathsf{JetCube}^{\boxtimes}_{\mathsf{Boo}}(\mathsf{fbe}, \vec{a})$ with $\boxtimes \in \{\square, \boxslash\}$, let $V$ be a jet cube with only $i$-directed variables called (from left to right) $\mathbf{j}_1, \ldots, \mathbf{j}_n$, and consider $\varphi : V \to (\mathbf{i} : (\!|P_i|\!))$ with $P \in \{\rightarrowtriangle, \leftarrowtriangle\}$. Then $\mathbf{i}\langle\varphi\rangle$ is either a constant or of the (obviously affine) form

$$\mathbf{i}\langle\varphi\rangle = (\ldots((\neg^{p_1}\mathbf{j}_1 \diamondsuit_1 \neg^{p_2}\mathbf{j}_2) \diamondsuit_2 \neg^{p_3}\mathbf{j}_3) \ldots) \diamondsuit_{n-1} \neg^{p_n}\mathbf{j}_n$$

with $p_k \in \{0,1\}$ and $\diamondsuit_k \in \{\vee, \wedge, \mathsf{K}\}$ where we define $x \mathrel{\mathsf{K}} y := y$.

We will only use this lemma when $\boxdot = \square$.

*Proof.* We assume $P = \rightarrowtriangle$, the proof for $P = \leftarrowtriangle$ is analogous.

We prove this by induction on $n$. If $n = 0$ then $\mathbf{i}\langle\varphi\rangle$ is necessarily a constant. Assume $n > 0$, implying that $a_i = \nearrow$. The jet set $\mathsf{JEP}(V)$ has $2^n$ elements and a unique Hamiltonian path of $i$-jets. The function $f := \mathsf{JEP}(\varphi)$ sends this Hamiltonian path to a path in $\mathsf{JEP}(\mathbf{i} : (\!|\rightarrowtriangle_i|\!)) = \{0 \rightarrowtriangle_i 1\}$. Thus, $f$ is entirely determined by the step in the Hamiltonian path where the image of $f$ flips from 0 to 1. Write $\mathbf{j}'_k$ to mean $\mathbf{j}_k$ if $\mathbf{j}_k : (\!|\rightarrowtriangle_i|\!)$ and to mean $\neg\mathbf{j}_k$ if $\mathbf{j}_k : (\!|\leftarrowtriangle_i|\!)$. There are 5 possible scenarios:

- The entire path is sent to 0. Then $\mathbf{i}\langle\varphi\rangle = 1 = \_ \mathrel{\mathsf{K}} 0$.
- The entire path is sent to 1. Then $\mathbf{i}\langle\varphi\rangle = 1 = \_ \mathrel{\mathsf{K}} 1$.
- The first half of the path is sent to 0, the second half is sent to 1. Then $\mathbf{i}\langle\varphi\rangle = \mathbf{j}'_n = \_ \mathrel{\mathsf{K}} \mathbf{j}'_n$.
- The output of $f$ flips somewhere in the first half of the path. Then $\mathbf{i}\langle\varphi\rangle = s \vee \mathbf{j}'_n$ for some boolean expression $s$ depending on $\mathbf{j}_1, \ldots, \mathbf{j}_{n-1}$. Write $V = (U, \mathbf{j}_n : \_)$. Then we have $(s/\mathbf{i}) : \mathsf{Op}^{\boxdot}_i U \to (\mathbf{i} : (\!|\rightarrowtriangle_i|\!))$, such that $\mathsf{JEP}(s/\mathbf{i})$ is essentially the restriction of $f$ to the first half of the Hamiltonian path as is evident from the following commutative diagram:

$$\begin{tikzcd}
\mathsf{Op}^{\boxdot}_i U \arrow[rr, "(s/\mathbf{i})"] \arrow[dr, "(0/\mathbf{j}_n)"'] & & (\mathbf{i} : (\!|\rightarrowtriangle_i|\!)). \\
& V = (U, \mathbf{j}_n : _) \arrow[ur, "(s \vee \mathbf{j}_n/\mathbf{i})"'] &
\end{tikzcd}$$

  By the induction hypothesis, $s$ is of the prescribed form, and therefore so is $\mathbf{i}\langle\varphi\rangle$.
- The output of $f$ flips somewhere in the second half of the path. Then $\mathbf{i}\langle\varphi\rangle = s \wedge \mathbf{j}'_n$ for some boolean expression $s$ depending on $\mathbf{j}_1, \ldots, \mathbf{j}_{n-1}$. Write $V = (U, \mathbf{j}_n : \_)$. Then we have $(s/\mathbf{i}) : U \to (\mathbf{i} : (\!|\rightarrowtriangle_i|\!))$, such that $\mathsf{JEP}(s/\mathbf{i})$ is essentially the restriction of $f$ to the second half of the Hamiltonian path as is evident from the following commutative diagram:

$$\begin{tikzcd}
U \arrow[rr, "(s/\mathbf{i})"] \arrow[dr, "(1/\mathbf{j}_n)"'] & & (\mathbf{i} : (\!|\rightarrowtriangle_i|\!)). \\
& V = (U, \mathbf{j}_n : _) \arrow[ur, "(s \wedge \mathbf{j}_n/\mathbf{i})"'] &
\end{tikzcd}$$

  By the induction hypothesis, $s$ is of the prescribed form, and therefore so is $\mathbf{i}\langle\varphi\rangle$. $\square$

**Lemma 1.4.3°23.** In $\mathsf{JetCube}^{\square}_{\mathsf{Boo}}(\mathsf{fbe}, \vec{a})$ where $a_i = \nearrow$, consider $\varphi : V \to (\mathbf{i} : (\!|P_i|\!))$ with $P \in \{\rightarrowtriangle, \leftarrowtriangle\}$. Write $\mathbf{j}_1, \ldots, \mathbf{j}_n$ for the $i$-directed variables of $V$. Then $\varphi$ does not depend on variables of degree lower (stronger) than $i$, nor on $i$-equijet variables of $V$. Moreover, $\mathbf{i}\langle\varphi\rangle$ is of the form

$$\mathbf{i}\langle\varphi\rangle = h_n(h_{n-1}(\ldots h_3(h_2(h_1(\neg^{p_1}\mathbf{j}_1) \diamondsuit_1 \neg^{p_2}\mathbf{j}_2) \diamondsuit_2 \neg^{p_3}\mathbf{j}_3) \ldots) \diamondsuit_{n-1} \neg^{p_n}\mathbf{j}_n)$$

with $p_k \in \{0,1\}$ and $\diamondsuit_k \in \{\vee, \wedge, \mathsf{K}\}$ where we define $x \mathrel{\mathsf{K}} y := y$, and every $h_k$ is a composition of functions of the form $\sqcup \heartsuit t$ with $t$ any affine boolean expression mentioning only $i$-symmetric variables and $\heartsuit \in \{\vee, \wedge, \mathsf{K}\}$.

*Proof.* We assume $P = \rightarrowtriangle$, the proof for $P = \leftarrowtriangle$ is analogous.

First of all, the $i$-equijet relation as well as all $\ell$-jet relations for $\ell < i$ are reflexive in $\mathsf{JEP}(\mathbf{i} : (\!|\rightarrowtriangle_i|\!)) = \{0 \rightarrowtriangle_i 1\}$, so that $\mathsf{JEP}(\varphi)$ must be constant on $i$-equijet- or $\ell$-jet-connected components, implying that $\varphi$ cannot depend on those variables. Then $\varphi$ factors over the map $\chi : V \to W$ that weakens over all those variables. ($\mathsf{JEP}(\chi)$ is the map that quotients out the $i$-equijet relation and therefore also the $\ell$-jet relations for $\ell < i$.) Thus, without loss of generality, we can assume that $V = W$ contains no variables of degree lower than $i$, and no $i$-equijet variables. Moreover, applying corollary 1.4.3°2, we can assume without loss of generality that $V$ is conventional so that the $i$-directed variables in $V$ are the last ones.

Write $U$ for the $i$-directed part of $V$ and note that any assignment of bits to all $i$-symmetric variables yields a jet cube morphism $\sigma : U \to V$ whose composite with $\varphi$ necessarily satisfies lemma 1.4.3°22. By lemma 1.4.3°10, there exists a particular assignment $\sigma_0 : U \to V$ such that $\mathbf{i}\langle\varphi \circ \sigma_0\rangle$ depends on all $i$-directed variables that $\mathbf{i}\langle\varphi\rangle$ depends on. The form of lemma 1.4.3°22 then dictates that $\mathbf{i}\langle\varphi\rangle$ depends on a final segment of the $i$-directed variables in $V$. Then $\varphi$ factors over the map $\chi : V \to W$ that weakens over the initial segment of $i$-directed variables that $\varphi$ does not depend on, and again, without loss of generality, we assume that $V = W$, i.e. that $\varphi$ uses *all* $i$-directed variables in $V$.

The fact that $\mathbf{i}\langle\varphi\circ\sigma_0\rangle$ satisfies lemma 1.4.3°22 and is a pruning of $\mathbf{i}\langle\varphi\rangle$ (by proposition 1.4.3°8), severely constrains the possible forms that $\mathbf{i}\langle\varphi\rangle$ may take. However, lemma 1.4.3°22 does not require that conjunction and disjunction appear in alternation, and thus, a priori, by associativity, parentheses could be moved around before unpruning. Thus, we need to constrain further to obtain the form in the current lemma.

Let $e$ be the unique normal affine representant of $\mathbf{i}\langle\varphi\rangle$ (theorem 1.4.3°20). We proceed by induction on $e$. If $e$ is a constant or $i$-symmetric literal, then it is of the required form. If it is an $i$-directed literal, then it is the only one it depends on, hence $\neg^{p_n}\mathbf{j}'_n$ with $n = 1$, and thus of the required form.

If $e$ is of the form $e = d_1 \diamondsuit \dots \diamondsuit d_m$, then we need to prove that either $\neg^{p_n}\mathbf{j}_n$ is a direct operand of $e$, or all $i$-directed variables occur in the same direct operand of $e$. Suppose that neither is the case. Then we can assume that $d_1$ depends *indirectly* on $\mathbf{j}_n$ and $d_2$ depends on $\mathbf{j}_k$. Say $d_1 = c_1 \clubsuit \dots \clubsuit c_\ell$ with $\{\diamondsuit, \clubsuit\} = \{\vee, \wedge\}$ and $c_1$ depends on $\mathbf{j}_n$. There is an assignment $\sigma_1$ of the $i$-symmetric dependencies of $d_1$ that reduces all $c_{i>2}$ to the absorbing element $\infty_\clubsuit$ of $\clubsuit$, and thus reduces $d_1$ to $\infty_\clubsuit = \iota_\diamondsuit$, the neutral element of $\diamondsuit$. For $i > 2$, there is an assignment $\sigma_i$ of the $i$-symmetric dependencies of $d_i$ that preserves all $i$-directed dependencies. We combine all these assignments to a single assignment $\sigma : U \to V$. Then $e[\sigma]$ reduces to a normal expression that depends on $\mathbf{j}_k$ but not $\mathbf{j}_n$, which is then a representant of $\mathbf{i}\langle\varphi\circ\sigma\rangle$, which we know must satisfy lemma 1.4.3°22 and thus cannot depend on $\mathbf{j}_k$ without also depending on $\mathbf{j}_n$. □

**Lemma 1.4.3°24.** In $\mathsf{JetCubeConv}^{\square}_{\mathsf{Boo}}(\mathsf{fbe}, \vec{a})$, let $\hat{\varphi} : V \to W$ be a jet cube morphism and write $\varphi = \lfloor\hat{\varphi}\rfloor$. Let $W = (W_0, \mathbf{i} : (\!|Q_i|\!))$ with $Q \in \{\rightarrowtriangle, \leftarrowtriangle\}$ and $a_i = \times\!\!\!\nearrow$. Let $e$ be the normal affine representant of $\mathbf{i}\langle\varphi\rangle$ and let $\mathbf{j}_1, \dots, \mathbf{j}_n$ ($n \geq 0$) be all the variables of degree $i$ that $e$ depends on, and $\mathbf{k}_1, \dots, \mathbf{k}_m$ ($m \geq 0$) all the other variables that $e$ depends on. Assume $m + n \geq 2$, so that $e$ necessarily contains a conjunction or disjunction. By lemma 1.4.3°23, we know that $\mathbf{j}_1, \dots, \mathbf{j}_n$ are the *last* $n$ variables of degree $i$ in $V$. Write $V = (V_0, \mathbf{j}_1 : (\!|P^1_i|\!), \dots, \mathbf{j}_n : (\!|P^n_i|\!), V_1)$ so that (even if $n = 0$) all variables in $V_0$ have degree at least (at the strongest) $i$ and all variables in $V_1$ have degree strictly less (stronger) than $i$. Here, each $P^1, \dots, P^n \in \{\rightarrowtriangle, \leftarrowtriangle\}$.[8] Define $\tilde{V} = (\mathsf{SymCl}^{\boxdot}_i V_0, \mathbf{j}_1 : (\!|P^1_i|\!), \dots, \mathbf{j}_n : (\!|P^n_i|\!), V_n)$, i.e. every variable of degree $i$ to the left of $\mathbf{j}_1$ gets promoted to an equijet variable. Then $\varphi$ is a jet cube morphism $\tilde{V} \to W$.

It is clear that $\mathbf{k}_1, \dots, \mathbf{k}_m$ all occur in $V_0$ as they cannot have degree less (stronger) than $i$.

In words, this lemma says that if the last variable $\mathbf{i}$ of $W$ is substituted with an expression $e$ depending on at least two variables, then all variables in $V$ of same degree as $\mathbf{i}$ that $e$ does *not* depend on, can be promoted to equijet variables.

*Proof.* For $c \in \{0, 1\}$, let $A_c$ be the set of all $(\vec{\kappa}, \vec{\zeta}) \in \{0, 1\}^{m+n}$ such that $\mathbf{i}\langle\varphi\rangle\langle\vec{\kappa}/\vec{\mathbf{k}}, \vec{\zeta}/\vec{\mathbf{j}}\rangle = c$. Let $U$ be the (ordinary) cube obtained from $\lfloor V \rfloor$ by removing all dependencies of $e$. Then for any $(\vec{\kappa}, \vec{\zeta}) \in \{0, 1\}^{m+n}$, by applying cube opposite functors in all the right places, there is a jet cube $U_{(\vec{\kappa}, \vec{\zeta})}$ such that $\left\lfloor U_{(\vec{\kappa}, \vec{\zeta})} \right\rfloor = U$ and $(\vec{\kappa}/\vec{\mathbf{k}}, \vec{\zeta}/\vec{\mathbf{j}}) : U_{(\vec{\kappa}, \vec{\zeta})} \to V$ is a jet cube morphism.

Then in the category of *jet cubes* and *cube morphisms between their erasures*, for any $(\vec{\kappa}, \vec{\zeta}) \in A_c$, we obtain a commutative diagram

$$
\begin{tikzcd}
U_{(\vec{\kappa}, \vec{\zeta})} \arrow[r, dotted, "\chi"] \arrow[d, "(\vec{\kappa}/\vec{\mathbf{k}}, \vec{\zeta}/\vec{\mathbf{j}})"'] & (\mathsf{Op}^{\boxdot}_i)^{1-c}(W_0) \arrow[d, "(c/\mathbf{i})"] \\
V \arrow[r, "\varphi"'] & W
\end{tikzcd}
$$

[8] We could in principle allow $\leftrightarrow$ but it is easy to see that $\varphi$ being a jet cube morphism (equivalently, $\mathsf{EP}(\varphi)$ being a jet set) implies $P^1, \dots, P^n \in \{\rightarrowtriangle, \leftarrowtriangle\}$.

where all the black lines are jet cube morphisms and the cube morphism $\chi$ is defined as $(\mathbf{i}/\oslash) \circ \varphi \circ (\vec{\kappa}/\vec{\mathbf{k}}, \vec{\zeta}/\vec{\mathbf{j}})$, which thanks to afineness does not depend on our choice of $(\vec{\kappa}, \vec{\zeta})$, nor even on $c$. Commutativity of the diagram and the fact that $\mathsf{JEP}(c/\mathbf{i})$ is a full jet set morphism, imply that $\chi$, too, is a jet cube morphism.

We now show that $\varphi$ is a jet cube morphsim $\tilde{V} \to W$. Let $\vec{v} \rightarrowtriangle_k \vec{v}'$ in $\tilde{V}$. We prove that $f(\vec{v}) \rightarrowtriangle_k f(\vec{v}')$ where $f = \mathsf{EP}(\varphi)$. If $\vec{v} = \vec{v}'$ then this is trivial, so let $\mathbf{l}$ be the variable where they differ. If $k \neq i$ or $\mathbf{l} \notin V_0$ then we have $\vec{v} \rightarrowtriangle_k \vec{v}'$ in $V$ so this is preserved by $f$.

Assume $k = i$ and $\mathbf{l} \in V_0$, which implies that $\mathbf{l}$ is not a dependency of $e$. This implies that $e\langle\vec{v}\rangle = e\langle\vec{v}'\rangle =: c$, or differently put $\mathbf{i}\langle f(\vec{v})\rangle = \mathbf{i}\langle f(\vec{v}')\rangle = c$. We have $\vec{v} \rightleftarrows_i \vec{v}'$ in $V$, say $\vec{v}\ S_i\ \vec{v}'$ where $S \in \{\rightarrowtriangle, \leftarrowtriangle\}$. This is preserved by $f$, so $f(\vec{v})\ S_i\ f(\vec{v}')$. Because $(c/\mathbf{i})$ is a full jet set morphism, writing $p = \mathsf{EP}(\mathbf{i}/\oslash)$, we can conclude that $p(f(\vec{v}))\ S_i\ p(f(\vec{v}'))$ in $(\mathsf{Op}_i^{\boxdot})^{1-c}(W_0)$.

Let $\vec{u}$ and $\vec{u}'$ be bit-assignments to the variables in $U$ obtained by projecting out all bits assigned to the dependencies of $e$ in the vectors $\vec{v}$ and $\vec{v}'$, and let $\vec{\zeta}$ and $\vec{\kappa}$ be the bits thus forgotten (which are the same for $\vec{v}$ and $\vec{v}'$). Thus, $\vec{v} = \mathsf{EP}(\vec{\zeta}/\vec{\mathbf{j}}, \vec{\kappa}/\vec{\mathbf{k}})(\vec{u})$ and similar for $\vec{v}'$. Writing $g = \mathsf{EP}(\chi)$, this implies that $p(f(\vec{v})) = g(\vec{u})$ and $p(f(\vec{v}')) = g(\vec{u}')$. Thus, we have $g(\vec{u})\ S_i\ g(\vec{u}')$ in $(\mathsf{Op}_i^{\boxdot})^{1-c}(W_0)$.

If $\varphi$ does not depend on $\mathbf{l}$, then we have nothing to prove, so let $\mathbf{h}$ be the variable of $W$ such that $\mathbf{h}\langle\varphi\rangle$ depends on $\mathbf{l}$. Note that $\mathbf{h} \neq \mathbf{i}$. We remark that the direction of the jet between $g(\vec{u})$ and $g(\vec{u}')$ in $(\mathsf{Op}_i^{\boxdot})^{1-c}(W_0)$ flips with $c = e\langle\vec{v}\rangle$, which is a function of $(\vec{\kappa}, \vec{\zeta})$.

On the other hand, looking at the direction of the jet between $\vec{u}$ and $\vec{u}'$ in $U_{(\vec{\kappa},\vec{\zeta})}$, we see that this flips with the $\mathbf{j}_1 \veebar \ldots \veebar \mathbf{j}_n$, the exclusive disjunction of all dependencies of $e$ of degree $i$, which all appear to the right of $\mathbf{l}$. Now if $m + n \geq 2$, then it is impossible that the affine boolean expression $e \in \mathsf{Boo}^{\#}(\{\mathbf{k}_1, \ldots, \mathbf{k}_m, \mathbf{j}_1, \ldots, \mathbf{j}_n\})$ which is in a reduced state and therefore truly depends on each of the mentioned variables, yields the exact same (or opposite) truth table as $\mathbf{j}_1 \veebar \ldots \veebar \mathbf{j}_n$.

Thus, fixing $\vec{u}$ and $\vec{u}'$ and varying $(\vec{\kappa}, \vec{\zeta})$, we see that there are assignments $(\vec{\kappa}, \vec{\zeta})$ for which the jets between $\vec{u}$ and $\vec{u}'$ in $U_{(\vec{\kappa},\vec{\zeta})}$ on one hand, and between $g(\vec{u})$ and $g(\vec{u}')$ in $(\mathsf{Op}_i^{\boxdot})^{1-c}(W_0)$ are aligned, and others for which they are opposed. Pick an assignment $(\vec{\kappa}', \vec{\zeta}')$ for which they are opposed. We also know that $\chi$ is a jet cube morphism for any assignment $(\vec{\kappa}, \vec{\zeta})$, and in particular for $(\vec{\kappa}', \vec{\zeta}')$. Thus, $\chi$ provides us the jet pointing the other way, and we can conclude that $g(\vec{u}) \leftrightarrowtriangle_i g(\vec{u}')$. Composing with $\mathsf{EP}(c/\mathbf{i})$ for our original $c$ yields $f(\vec{v}) \leftrightarrowtriangle_i f(\vec{v}')$. □

**Corollary 1.4.3°25.** In $\mathsf{JetCube}^{\square}_{\mathsf{Boo}}(\mathsf{fbe}, \vec{a})$, let $\hat{\varphi} : V \to W$ be a jet cube morphism and write $\varphi = \lfloor\hat{\varphi}\rfloor$. Let $W = (W_0, \mathbf{i} : (\!|Q_i|\!))$ with $Q \in \{\rightarrowtriangle, \leftarrowtriangle\}$ and $a_i = \nearrow$. Let $e$ be the reduction of $\mathbf{i}\langle\varphi\rangle$ and assume $e$ depends on at least two variables. Then $(\oslash/\mathbf{i}) \circ \varphi$ is a jet cube morphism $\mathsf{SymCl}_i^{\boxdot} V \to W_0$.

*Proof.* We know from lemma 1.4.3°24 that $\varphi$ is a jet cube morphism $\tilde{V} \to W$. We first show that $(\oslash/\mathbf{i}) \circ \varphi$ is a jet cube morphism $\tilde{V} \to W_0$. Write $f = \mathsf{EP}(\varphi)$ and $p = \mathsf{EP}(\oslash/\mathbf{i})$. Pick a non-reflexive jet $\vec{v} \rightarrowtriangle_j \vec{v}'$ in $\tilde{V}$. We show that $p(f(\vec{v})) \rightarrowtriangle_j p(f(\vec{v}'))$ in $W_0$. We know that $f(\vec{v}) \rightarrowtriangle_j f(\vec{v}')$ in $W$. If $j \neq i$, it follows that $p(f(\vec{v})) \rightarrowtriangle_j p(f(\vec{v}'))$ in $W_0$. If $j = i$, it follows that $p(f(\vec{v})) \rightleftarrows_i p(f(\vec{v}'))$ in $W_0$. But because $(\oslash/\mathbf{i}) \circ \varphi$ only depends on $i$-symmetric variables in $\tilde{V}$, we can conclude that $p(f(\vec{v})) \leftrightarrowtriangle_i p(f(\vec{v}'))$.

We conclude that $(\oslash/\mathbf{i}) \circ \varphi$ is a jet cube morphism $\tilde{V} \to W$. Since it only depends on $i$-symmetric variables, there is no harm in promoting the ignored variables of degree $i$ to equijet variables, and that is all that $\mathsf{SymCl}_i^{\boxdot} V$ does. □

### 1.4.3 (c) Completeness

**Theorem 1.4.3°26** (Completeness)**.** For $M \in \{\mathsf{IPt}_2, \mathsf{Boo}\}$ and any morphism $\hat{\varphi} : V \to W$ in $\mathsf{JetCubeConv}^{\square}_M(\mathsf{fbe}, \vec{a})$, writing $\varphi = \lfloor\hat{\varphi}\rfloor$, we have $\vdash \varphi : V \to W$.

*Proof.* For each variable $\mathbf{k}$ in $W$, let $e_{\mathbf{k}}$ be the normal affine representant of $\mathbf{k}\langle\varphi\rangle$.

We prove completeness by induction on the number of nodes and leaves (added up) in the tuple $(e_{\mathbf{k}})_{\mathbf{k}\in W}$, plus the number of directed degrees in the mask.

If $W = ()$, then use TERMINAL.

If $a_i = \cancel{\nearrow}$ and the last variable in $W$ is $\mathbf{i} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis$, then $W$ is of the form $\mathsf{USym}_i^{\boxdot}(U)$ and we can use the induction hypothesis for the corresponding morphism $\mathsf{FSym}_i^{\boxdot}(V) \to U$ and we can use the rule SYMMETRIZE.

In the remaining case, the last variable in $W$ is not an equijet dimension at a directed degree, i.e. it is of the form $\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis$ or $\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis$.

If the last variable in $V$ is of degree strictly lower (stronger) than the degree of $\mathbf{i}$, then in order to be a jet set morphism, $\mathsf{JEP}(\hat{\varphi})$ cannot depend on that variable, so we can invoke WKN until the last variable in $V$ is of degree at least $i$. We do not resort to the induction hypothesis but proceed below.

We proceed by inspecting $e_{\mathbf{i}}$.

- If $[\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis$ and $e_{\mathbf{i}} = 0]$ or $[\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis$ and $e_{\mathbf{i}} = 1]$, then $\varphi$ being a jet cube morphism $V \to W = (U, \mathbf{i} : \_)$ (equivalently: $\mathsf{EP}(\varphi)$ being a jet set morphism $\mathsf{JEP}(V) \to \mathsf{JEP}(W)$) is equivalent to $(\oslash/\mathbf{i}) \circ \varphi$ being a jet cube morphism $V \to \mathsf{Op}_i^{\boxdot}(U)$, so we can apply SRC:FWD or SRC:BCK.
- If $[\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis$ and $e_{\mathbf{i}} = 1]$ or $[\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis$ and $e_{\mathbf{i}} = 0]$, then $\varphi$ being a jet cube morphism $V \to W = (U, \mathbf{i} : \_)$ (equivalently: $\mathsf{EP}(\varphi)$ being a jet set morphism $\mathsf{JEP}(V) \to \mathsf{JEP}(W)$) is equivalent to $(\oslash/\mathbf{i}) \circ \varphi$ being a jet cube morphism $V \to U$, so we can apply TGT:FWD or TGT:BCK.
- If $\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis$ and $e_{\mathbf{i}} = \neg\mathbf{j}$, then $\varphi$ being a jet cube morphism $V \to W = (U, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$ is equivalent to $(\neg\mathbf{i}/\mathbf{i}) \circ \varphi$ being a jet cube morphism $V \to (U, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis)$, so we can apply INV:FWD. Similarly, if $\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis$ and $e_{\mathbf{i}} = \neg\mathbf{j}$, we can apply INV:BCK.
- If $\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis$ ($\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis$ is handled analogously) and $e_{\mathbf{i}} = \mathbf{j}$ where $V$ specifies that $\mathbf{j}$ has degree $j$, then we know that $\varphi = (\chi, \mathbf{j}/\mathbf{i})$ is a jet cube morphism $V = (V_0, \mathbf{j} : \llparenthesis R_j \rrparenthesis, V_1) \to W = (U, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$ for some $R \in \{\rightarrowtriangle, \leftarrowtriangle, \leftrightarrowtriangle\}$. We have the following commutative diagram in the category of *jet cubes* and *cube morphisms between erased jet cubes*:

$$\begin{tikzcd}
(\mathsf{Op}_j^{\boxdot}(V_0), V_1) \arrow[r, dotted, "\chi"] \arrow[d, "(0/\mathbf{j})"'] & \mathsf{Op}_i^{\boxdot}(U) \arrow[d, "(0/\mathbf{i})"] \\
(V_0, \mathbf{j} : \llparenthesis R_j \rrparenthesis, V_1) \arrow[r, "\varphi = (\chi, \mathbf{j}/\mathbf{i})"'] & (U, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \\
(V_0, V_1) \arrow[r, dotted, "\chi"'] \arrow[u, "(1/\mathbf{j})"] & U \arrow[u, "(1/\mathbf{i})"']
\end{tikzcd}$$

We know that the black arrows are all jet cube morphisms, and the vertical arrows all yield full jet set morphisms (definition 1.2.1°2). This implies that the dashed arrows also lift to jet set morphisms, i.e. are jet cube morphisms. Then $\chi$ is both a jet cube morphism $(V_0, V_1) \to U$ and $\mathsf{Op}_i^{\boxdot}(\mathsf{Op}_j^{\boxdot}(V_0), V_1) \to U$.

  - If $j = i$, then all variables in $V_1$ have degree $i$ and $\chi$ is both a jet cube morphism $(V_0, V_1) \to U$ and $\mathsf{Op}_i^{\boxdot}(\mathsf{Op}_i^{\boxdot} V_0, V_1) \to U$. This implies that $\mathsf{EP}(\varphi)$ sends every $i$-jet of the form $(\vec{v}_0, r, \vec{v}_1) \rightarrowtriangle_i (\vec{v}_0, r, \vec{v}_1')$ in $\mathsf{JEP}(V_0, V_1)$ – which points the other way in $\mathsf{JEP}(\mathsf{Op}_i^{\boxdot}(\mathsf{Op}_i^{\boxdot} V_0, V_1))$ – to an $i$-equijet $(\mathsf{EP}(\chi)(\vec{v}_0, \vec{v}_1), r) \leftrightarrowtriangle_i (\mathsf{EP}(\chi)(\vec{v}_0, \vec{v}_1'), r)$ in $\mathsf{JEP}(U, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$.
    * If $a_i = \cancel{\nearrow}$, then for any bit-assignment $\vec{v}_1$ of the variables in $V_1$, we get a jet cube morphism $(\chi \circ (\vec{v}_1/V_1), \mathbf{j}/\mathbf{i}) : (\mathsf{Op}_i^{\boxdot})^p(V_0, \mathbf{j} : \llparenthesis R_i \rrparenthesis) \to (U, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$, where $p$ is the number of source constants in $\vec{v}$ (i.e. 0 for $\llparenthesis \rightarrowtriangle_i \rrparenthesis$ and 1 for $\llparenthesis \leftarrowtriangle_i \rrparenthesis$). This implies that $p$ is the same for all assignments $\vec{v}$, which is only possible if $V_1 = ()$. In that case, it is easy to see that $R = \rightarrowtriangle$. Thus, we can apply PRISM:FWD.
    * If $a_i = \bigcirc$, then we can use EXCHANGE to create a morphism from $(V_0, V_1, \mathbf{j} : \llparenthesis \frown_i \rrparenthesis)$ instead, which can be done using PRISM:FWD (or equivalently PRISM:BCK).

  If $j > i$, then $\chi$ is necessarily a jet cube morphism $\mathsf{SymCl}_i^{\boxdot}(\mathsf{SymCl}_j^{\boxdot}(V_0), V_1) \to U$, so we can apply CONCURSOR.

- We have now covered all cases for the monad $\mathsf{IPt}_2$. In the remaining cases, $e_{\mathbf{i}}$ contains connection (conjunction or disjunction) symbols. If $\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis$, we apply INV:FWD and push down the introduced negation, after which we do not resort to the induction hypothesis but proceed below.[9] We now assume that $\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis$.

- We first treat the case where $a_i = \bigcirc$. Let $\varphi = (\chi, s \diamondsuit t/\mathbf{i})$ where $\diamondsuit \in \{\vee, \wedge\}$ and $W = (U, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$. We claim that if $\varphi$ is a jet cube morphism $V \to W = (U, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis)$, then so are $(\chi, s/\mathbf{i})$ and $(\chi, t/\mathbf{i})$, so that we can invoke CONN:SYM. Write

  $$f = \mathsf{EP}(\varphi), \qquad g = \mathsf{EP}(\chi), \qquad p = \mathsf{EP}(\chi, s/\mathbf{i}), \qquad q = \mathsf{EP}(\chi, t/\mathbf{i}).$$

  Pick a non-reflexive jet $\vec{v} \rightarrowtriangle_j \vec{v}'$. We prove $p(\vec{v}) \rightarrowtriangle_j p(\vec{v}')$; by symmetry of the situation we do not have to prove the same for $q$. Let $\mathbf{k}$ be the variable of $V$ where $\vec{v}$ and $\vec{v}'$ differ. There are four possible situations:
  - If $\varphi$ does not depend on $\mathbf{k}$, then we are done.
  - If $\chi$ depends on $\mathbf{k}$, then we have $s\langle\vec{v}\rangle = s\langle\vec{v}'\rangle =: s_0$ and $t\langle\vec{v}\rangle = t\langle\vec{v}'\rangle =: t_0$.
    - If $j \neq i$, then from $(g(\vec{v}), s_0 \diamondsuit t_0) = f(\vec{v}) \rightarrowtriangle_j f(\vec{v}') = (g(\vec{v}'), s_0 \diamondsuit t_0)$, it follows that $g(\vec{v}) \rightarrowtriangle_j g(\vec{v}')$ in $\mathsf{JEP}(U)$, whence $p(\vec{v}) = (g(\vec{v}), s_0) \rightarrowtriangle_j (g(\vec{v}'), s_0) = p(\vec{v}')$.
    - If $j = i$, then from $(g(\vec{v}), s_0 \diamondsuit t_0) = f(\vec{v}) \frown_i f(\vec{v}') = (g(\vec{v}'), s_0 \diamondsuit t_0)$ it follows that $g(\vec{v}) \frown_i g(\vec{v}')$ in $\mathsf{JEP}(U)$, whence $p(\vec{v}) = (g(\vec{v}), s_0) \frown_i (g(\vec{v}'), s_0) = p(\vec{v}')$.
  - If $s$ depends on $\mathbf{k}$, then we have $g(\vec{v}) = g(\vec{v}') =: g_0$ and $t\langle\vec{v}\rangle = t\langle\vec{v}'\rangle =: t_0$. Pick a bit-assignment $\tau$ of the dependencies of $t$ such that $t\langle\tau\rangle$ reduces to the neutral element $\iota_\diamondsuit$ of $\diamondsuit$.[10] Define $\vec{x}$ and $\vec{x}'$ by overwriting $\vec{v}$ and $\vec{v}'$ with $\tau$. Then $g(\vec{x}) = g(\vec{x}') = g_0$ and $t\langle\vec{x}\rangle = t\langle\vec{x}'\rangle = \iota_\diamondsuit$ and $s\langle\vec{x}\rangle = s\langle\vec{v}\rangle$ and $s\langle\vec{x}'\rangle = s\langle\vec{v}'\rangle$. We have $\vec{x} \rightleftarrows_j \vec{x}'$, whence

    $$\begin{array}{ccccccccc} p(\vec{v}) & = & (g_0, s\langle\vec{v}\rangle) & = & (g_0, s\langle\vec{v}\rangle \diamondsuit \iota_\diamondsuit) & = & (g_0, s\langle\vec{x}\rangle \diamondsuit t\langle\vec{x}\rangle) & = & f(\vec{x}) \\ & & & & & & & & \rightleftarrows_j \\ p(\vec{v}') & = & (g_0, s\langle\vec{v}'\rangle) & = & (g_0, s\langle\vec{v}'\rangle \diamondsuit \iota_\diamondsuit) & = & (g_0, s\langle\vec{x}'\rangle \diamondsuit t\langle\vec{x}'\rangle) & = & f(\vec{x}') \end{array}$$

    in $\mathsf{JEP}(W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$. Now, since these vectors only differ at $\mathbf{i}$, we can deduce equality if $j < i$ ($j$ is stronger than $i$) and otherwise that $p(\vec{v}) \frown_i p(\vec{v}')$, which implies $p(\vec{v}) \rightarrowtriangle_j p(\vec{v}')$ since $j \geq i$ ($j$ is weaker than or equal to $i$).
  - If $t$ depends on $\mathbf{k}$, then $\chi$ and $s$ do not so $p(\vec{v}) = p(\vec{v}')$ and we are done.
- Now assume that $a_i = \nearrow$. Let $\varphi = (\chi, t \diamondsuit s/\mathbf{i})$ where $\diamondsuit \in \{\vee, \wedge\}$ and $W = (U, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$. Corollary 1.4.3°25 immediately tells us that $\chi$ is a jet cube morphism $\mathsf{SymCl}^{\boxdot} V \to U$. By lemma 1.4.3°23, we can assume that $s$ is either (the negation of) the last variable of degree $i$ in $V = (V_0, \mathbf{i} : \llparenthesis P_i \rrparenthesis)$ with $P \in \{\rightarrowtriangle, \leftarrowtriangle\}$ or a boolean expression only depending on $i$-symmetric variables.
  - If $s = \mathbf{i}' \in \{\mathbf{i}, \neg\mathbf{i}\}$, then depending on $\diamondsuit$ we have one of the following commutative diagrams, where each arrow is a jet cube morphism but labelled with its erasure:

    $$\begin{tikzcd} & \mathsf{Op}_i^{\boxdot} V \arrow[d, "(0/\mathbf{i})"'] \arrow[dr, bend left, "(\chi, t/\mathbf{i})"] & \\ (V, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis) \arrow[r, leftrightarrow, "\cong", "(\chi, \neg\mathbf{i}/\mathbf{i})"'] & (V, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \arrow[r, "(\chi, t \vee \mathbf{i}/\mathbf{i})"'] & (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \end{tikzcd}$$

    $$\begin{tikzcd} & V \arrow[d, "(1/\mathbf{i})"'] \arrow[dr, bend left, "(\chi, t/\mathbf{i})"] & \\ (V, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis) \arrow[r, leftrightarrow, "\cong", "(\chi, \neg\mathbf{i}/\mathbf{i})"'] & (V, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \arrow[r, "(\chi, t \wedge \mathbf{i}/\mathbf{i})"'] & (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \end{tikzcd}$$

    Thus, we can invoke one of the rules CONN:PRISM:SRC-NEUTRAL, CONN:PRISM:TGT-NEUTRAL, CONN:PRISM-INV:SRC-NEUTRAL, CONN:PRISM-INV:TGT-NEUTRAL.

[9] Alternatively, we could duplicate and adapt the proof below to the case where $\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis$.

[10] If this were not possible, then $t$ would be a constant, which is in contradiction with the assumption that $e_\mathbf{i}$ was normal.

* If $s$ depends only on $i$-symmetric variables, then the same holds for $(\chi, s/\mathbf{i})$. Write

$$f = \mathsf{EP}(\varphi), \qquad g = \mathsf{EP}(\chi), \qquad p = \mathsf{EP}(\chi, s/\mathbf{i}), \qquad q = \mathsf{EP}(\chi, t/\mathbf{i}).$$

We show that

· $(\chi, s/\mathbf{i})$ is a jet cube morphism $\mathsf{SymCl}_i^{\boxdot} V \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$,
· $(\chi, t/\mathbf{i})$ is a jet cube morphism $V \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$,

so that we can invoke CONN:DEGREE-SYMMETRIC.
Pick a non-reflexive jet $\vec{v} \rightarrowtriangle_j \vec{v}'$ in $\mathsf{JEP}(V)$. We will prove $p(\vec{v}) \rightarrowtriangle_j p(\vec{v}')$ and $q(\vec{v}) \rightarrowtriangle_j q(\vec{v}')$ and, if $j = i$, even $p(\vec{v}) \leftrightarrowtriangle_j p(\vec{v}')$, all the time in $\mathsf{JEP}(W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$. Let $\mathbf{k}$ be the variable where $\vec{v}$ and $\vec{v}'$ differ. There are four possibilities:

· If $\varphi$ does not depend on $\mathbf{k}$, then we are done.
· If $\chi$ depends on $\mathbf{k}$, then $s\langle\vec{v}\rangle = s\langle\vec{v}'\rangle =: s_0$ and $t\langle\vec{v}\rangle = t\langle\vec{v}'\rangle =: t_0$.
  · If $j = i$, then $g(\vec{v}) \leftrightarrowtriangle_i g(\vec{v}')$ and hence $p(\vec{v}) = (g(\vec{v}), s_0) \leftrightarrowtriangle_i (g(\vec{v}'), s_0) = p(\vec{v}')$ and $q(\vec{v}) = (g(\vec{v}), t_0) \leftrightarrowtriangle_i (g(\vec{v}'), t_0) = q(\vec{v}')$ as required.
  · If $j \neq i$, then $g(\vec{v}) \rightarrowtriangle_j g(\vec{v}')$ and hence $p(\vec{v}) = (g(\vec{v}), s_0) \rightarrowtriangle_j (g(\vec{v}'), s_0) = p(\vec{v}')$ and $q(\vec{v}) = (g(\vec{v}), t_0) \rightarrowtriangle_j (g(\vec{v}'), t_0) = q(\vec{v}')$ as required.
· If $s$ depends on $\mathbf{k}$, then $g(\vec{v}) = g(\vec{v}') =: g_0$ and $t\langle\vec{v}\rangle = t\langle\vec{v}'\rangle =: t_0$, so $q(\vec{v}) = q(\vec{v}')$. We pick $\tau$ and define $\vec{x}$ and $\vec{x}'$ in the same way as we did when $a_i = \bigcirc$ and all other circumstances were the same. Then we have $\vec{x} \rightleftarrows_j \vec{x}'$ in $\mathsf{JEP}(V)$, whence $p(\vec{v}) = f(\vec{x}) \rightleftarrows_j f(\vec{x}') = p(\vec{v}')$. Now, since these vectors only differ at $\mathbf{i}$, we can deduce equality if $j < i$ ($j$ is stronger than $i$) and otherwise that $p(\vec{v}) \rightleftarrows_i p(\vec{v}')$. But since $s$ and $\chi$ only depend on $i$-symmetric variables, it must be the case that $p(\vec{v}) \leftrightarrowtriangle_i p(\vec{v}')$, as required if $i = j$, and implying $p(\vec{v}) \rightarrowtriangle_j p(\vec{v}')$ if $j > i$.
· If $t$ depends on $\mathbf{k}$, then $g(\vec{v}) = g(\vec{v}') =: g_0$ and $s\langle\vec{v}\rangle = s\langle\vec{v}'\rangle =: s_0$, so $p(\vec{v}) = p(\vec{v}')$. Pick an assignment $\sigma$ of the dependencies of $s$ such that $s\langle\sigma\rangle$ reduces to the neutral element $\iota_\diamondsuit$ of $\diamondsuit$. Define $\vec{y}$ and $\vec{y}'$ by overwriting $\vec{v}$ and $\vec{v}'$ with $\sigma$. Then $g(\vec{y}) = g(\vec{y}') = g_0$ and $s\langle\vec{y}\rangle = s\langle\vec{y}'\rangle = \iota_\diamondsuit$ and $t\langle\vec{y}\rangle = t\langle\vec{v}\rangle$ and $t\langle\vec{y}'\rangle = t\langle\vec{v}'\rangle$. We have $\vec{y} \rightleftarrows_j \vec{y}'$ but, since all dependencies of $s$ are $i$-symmetric, they come to the left of all $i$-directed variables, so if $i = j$ we actually still have $\vec{y} \rightarrowtriangle_i \vec{y}'$. Now we have

$$\begin{array}{ccccccccc}
q(\vec{v}) & = & (g_0, t\langle\vec{v}\rangle) & = & (g_0, \iota_\diamondsuit \diamondsuit t\langle\vec{v}\rangle) & = & (g_0, s\langle\vec{y}\rangle \diamondsuit t\langle\vec{y}\rangle) & = & f(\vec{y}) \\
 & & & & & & & & \rightleftarrows_j \\
q(\vec{v}') & = & (g_0, t\langle\vec{v}'\rangle) & = & (g_0, \iota_\diamondsuit \diamondsuit t\langle\vec{v}'\rangle) & = & (g_0, s\langle\vec{y}'\rangle \diamondsuit t\langle\vec{y}'\rangle) & = & f(\vec{y}')
\end{array}$$

and $q(\vec{v}) \rightarrowtriangle_i q(\vec{v}')$ if $j = i$. Thus, the case $j = i$ has been handled. Since $q(\vec{v})$ and $q(\vec{v}')$ can only differ at $\mathbf{i}$ which has degree $i$, we can deduce equality if $j < i$ ($j$ is stronger than $i$) and otherwise that $q(\vec{v}) \rightleftarrows_i q(\vec{v}')$ which implies $q(\vec{v}) \rightarrowtriangle_j q(\vec{v}')$ as required. $\square$

### 1.4.4 The Semisymmetric Separated Product

**Definition 1.4.4°1.** We define the **separated product** functor

$$\sqcup * \sqcup : \mathsf{JetSet}(\vec{a}) \times \mathsf{JetSet}(\vec{a}) \to \mathsf{JetSet}(\vec{a})$$

by letting $X * Y$ be the jet set with carrier $UX \times UY$ such that $(x, y) \rightarrowtriangle_j (x', y')$ if either

- $x \rightarrowtriangle_j x'$ and $y = y'$,
- $x = x'$ and $y \rightarrowtriangle_j y'$.

The action on morphisms is of course faithfully inherited from the cartesian product functor on Set, which indeed produces jet set morphisms between separated products.

**Definition 1.4.4°2.** Given masks $\vec{a}$ and $\vec{b}$ of equal length, if $\vec{a} \sqcap \vec{b} = \vec{\bigcirc}$, i.e. if for every $i$ we have $a_i = \bigcirc$ and/or $b_i = \bigcirc$, then we define the **semisymmetric separated product (SSS-product)** functor

$$\sqcup \S \sqcup : \mathsf{JetCubeConv}^{\text{⌑}}_M(\mathsf{fbe}, \vec{a}) \times \mathsf{JetCubeConv}^{\text{⌑}}_M\left(\mathsf{fbe}, \vec{b}\right) \to \mathsf{JetCubeConv}^{\text{⌑}}_M\left(\mathsf{fbe}, \vec{a} \sqcup \vec{b}\right)$$

as follows:

- The variables of a pair of objects $(V, W)$ are zipped on a per-degree basis:
  - If $a_i = b_i = \bigcirc$, then we list all variables of degree $i$ of $V$, followed by all variables of degree $i$ of $W$, all of them typed as $\llparenthesis \frown_i \rrparenthesis$,
  - If $a_i = \nearrow$ and $b_i = \bigcirc$, then we list all variables of degree $i$ of $W$, retyped as $\llparenthesis \leftrightarrow_i \rrparenthesis$, followed by all variables of degree $i$ of $V$ with their original types,
  - If $a_i = \bigcirc$ and $b_i = \nearrow$, then we list all variables of degree $i$ of $V$, retyped as $\llparenthesis \leftrightarrow_i \rrparenthesis$, followed by all variables of degree $i$ of $W$ with their original types.

  **Corollary 1.4.4°3.** In $\mathsf{Cube}^{\text{⌑}}_M$, we have

  $$\begin{aligned} \lfloor V \S W \rfloor &\cong \lfloor V \rfloor * \lfloor W \rfloor && \text{if } \text{⌑} = \Box, \\ \lfloor V \S W \rfloor &\cong \lfloor V \rfloor \times \lfloor W \rfloor && \text{if } \text{⌑} = \boxslash. \end{aligned}$$

  **Corollary 1.4.4°4.** In $\mathsf{JetSet}(\vec{a} \sqcup \vec{b})$, we have

  $$\mathsf{JEP}(V \S W) \cong \mathsf{USym}_{\vec{a} \sqsubseteq \vec{a} \sqcup \vec{b}}(V) * \mathsf{USym}_{\vec{b} \sqsubseteq \vec{a} \sqcup \vec{b}}(W),$$

  where $\mathsf{USym}_{\vec{x} \sqsubseteq \vec{y}} : \mathsf{JetSet}(\vec{x}) \to \mathsf{JetSet}(\vec{y})$ is the forgetful functor.

- Recalling definition 1.4.2°8, the action of morphisms is established as follows:
  - At the level of $\mathsf{Cube}^{\text{⌑}}_M$, by relying on functoriality of the separated/cartesian product,
  - At the level of $\mathsf{JetSet}(\vec{a} \sqcup \vec{b})$, by relying on functoriality of the separated product,
  - At the level of $\mathsf{Set}$, both of these approaches reduce to functoriality of the cartesian product.

By corollary 1.4.3°2, the SSS-product extends to non-conventional jet cubes.

**Definition 1.4.4°5.** Given a fixed length $\ell$, which we assume clear from the context, and a degree $0 \leq i < \ell$, we define the **punch mask** $\vec{\delta}^i$ by $\vec{\delta}^i_j = \bigcirc$ if $i \neq j$ and $\vec{\delta}^i_i = \nearrow$.

Thus, $\bigsqcup_i \vec{\delta}^i = \vec{\nearrow}$, and more generally $\vec{a} = \bigsqcup_i (\vec{\delta}^i \sqcap \vec{a})$.

**Theorem 1.4.4°6** (SSS-factorization)**.** Let $\vec{a}$ be a mask of length $\ell$. For any jet cube morphism $\hat{\varphi} : V \to W$ in $\mathsf{JetCubeConv}^{\Box}_M(\mathsf{fbe}, \vec{a})$, define jet cubes $(V_i)_{0 \leq i < \ell}$ and $(W_i)_{0 \leq i < \ell}$ of mask $\vec{a} \sqcap \vec{\delta}^i$ by

- $W_i$ consists of all variables of $W$ of degree $i$, with their original typing,
- $V_i$ as the jet cube consisting of all variables $\mathbf{j}$ of $V_i$ such that there is a variable $\mathbf{i} \in W_i$ such that $\mathbf{i}\langle\varphi\rangle$ depends on $\mathbf{j}$. Variables of degree $i$ are kept with their original typing, unless a variable of the same degree to their right has been used at a lower (stronger) degree or is not used at all, in which case they are retyped as $\llparenthesis \leftrightarrow_i \rrparenthesis$. Variables of higher (weaker) degree $j > i$ are retyped as $\llparenthesis \frown_j \rrparenthesis$. Variables of lower degree cannot occur.

Then there are jet cube morphisms $\varphi_i : V_i \to W_i$ and jet cube morphisms $\rho_0$ and $\rho_1$ that erase to cube renamings[11], such that $\varphi$ factorizes as:

$$V \xrightarrow{\ \rho_0\ } \prod^{\S}_i V_i \xrightarrow{\ \prod^{\S}_i \varphi_i\ } \prod^{\S}_i W_i \xrightarrow[\cong]{\ \rho_1\ } W,$$

where $\prod^{\S}$ denotes a semisymmetric separated product.

[11] Morphisms in $\mathsf{Kl}(M)$ that come from $\mathsf{Set}$, i.e. are not effectful or do not use the constants and operators provided by $M$.

*Proof.* It is immediately clear that $\prod_i^{\S} W_i \cong W$ by an isomorphism $\rho_1$ that erases to a renaming.[12]

We then define $\rho_0$ as the cube morphism that discards all variables unused by $\varphi$, and $\varphi_i$ as the cube morphism such that $\mathbf{i}\langle\varphi_i \circ \rho_0\rangle = \mathbf{i}\langle\varphi\rangle$ for every variable $\mathbf{i}$ in $W_i$. It is then immediately clear that $\varphi = \rho_1 \circ \left(\prod_i^{\S} \varphi_i\right) \circ \rho_0$. What remains to be proven is that $\rho_0$ and $\varphi_i$ are jet cube morphisms.

In the case of $\rho_0$, this is relatively trivial: we are promoting an initial segment of the variables of every degree to equijet variables of the same degree, and then projecting away some of these.

In the case of $\varphi_i$, this follows essentially from lemma 1.4.3°21. $\square$

### 1.4.5 Comparison to the Literature

#### 1.4.5 (a) Point category

**Proposition 1.4.5°1.** The point category (terminal category) is isomorphic to $\mathsf{JetCube}^{\lozenge}_M(\omega, [])$.

*Proof.* It is clear that $()$ is the only object. The only endomorphism of $()$ in $\mathsf{Cube}^{\lozenge}_M$ is the identity, and $\lfloor \sqcup \rfloor$ is faithful, so there is only one morphism. $\square$

#### 1.4.5 (b) Affine Symmetric Cubes

I am unsure whether the category of affine symmetric cubes $\mathsf{Cube}^{\square}_{\mathsf{IPt}_2}$ appears anywhere.

**Proposition 1.4.5°2.** The category $\mathsf{Cube}^{\square}_{\mathsf{IPt}_2}$ is isomorphic to $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\omega, [\bigcirc])$ and $\mathsf{JetCube}^{\boxslash}_{\mathsf{IPt}_2}(\omega, [\bigcirc])$.

*Proof.* The orientation kit $\omega$ does not matter as all degrees are symmetric. The latter two categories are isomorphic by theorem 1.4.2°15. It is clear that $\lfloor \sqcup \rfloor : \mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\omega, [\bigcirc]) \to \mathsf{Cube}^{\square}_{\mathsf{IPt}_2}$ is bijective on objects. It is faithful, because $U : \mathsf{JetSet}([\bigcirc]) \to \mathsf{Set}$ is faithful. It is full, because any morphism can be derived in the calculus (fig. 1.1), as can be shown by induction on the length of the codomain. $\square$

#### 1.4.5 (c) Affine Cubes

The category of affine cubes $\mathsf{Cube}^{\square}_{\mathsf{Pt}_2}$ appears in a cubical model of HoTT [BCH14] and its unary analogue in a cubical model of parametricity [BCM15].

**Proposition 1.4.5°3.** The category $\mathsf{Cube}^{\square}_{\mathsf{Pt}_2}$ is isomorphic to $\mathsf{JetCube}^{\square}_{\mathsf{Pt}_2}(\omega, [\bigcirc])$ and $\mathsf{JetCube}^{\boxslash}_{\mathsf{Pt}_2}(\omega, [\bigcirc])$.

*Proof.* Each time, the category for $\mathsf{Pt}_2$ is the wide[13] subcategory of the corresponding one for $\mathsf{IPt}_2$ on morphisms that do not mention $\neg$, so the result follows from proposition 1.4.5°2. $\square$

#### 1.4.5 (d) Cartesian Cubes

One might hope to retrieve other existing categories as follows:

- De Morgan cubes $\mathsf{Cube}^{\boxslash}_{\mathsf{DM}}$ [CCHM15] as $\mathsf{JetCube}^{\boxslash}_{\mathsf{DM}}(\mathsf{f}, [\bigcirc])$,
- Cartesian cubes $\mathsf{Cube}^{\boxslash}_{\mathsf{Pt}_2}$ as $\mathsf{JetCube}^{\boxslash}_{\mathsf{DM}}(\mathsf{f}, [\bigcirc])$,
- Depth $n$ cubes [ND18, Nuy18] as $\mathsf{JetCube}^{\boxslash}_{\mathsf{Pt}_2}(\mathsf{f}, [\bigcirc]^n)$, where $\vec{x}^n$ denotes the $n$-fold repetition of the list $\vec{x}$,
- …

However, one cannot:

**Proposition 1.4.5°4.** The morphism $(\mathbf{i}/\mathbf{j}, \mathbf{i}/\mathbf{k}) : (\mathbf{i} : \mathbb{I}) \to (\mathbf{j} : \mathbb{I}, \mathbf{k} : \mathbb{I})$ in $\mathsf{Cube}^{\boxslash}_M$ is not the erasure of any jet cube morphism.

*Proof.* Jet sets obtained from $\mathsf{JEP}$ do not have diagonals. $\square$

[12] In fact $\rho_1$ is the identity because we are working with conventional cubes.
[13] Containing all objects.

### 1.4.5 (e) Affine Depth $n$ Cubes

**Definition 1.4.5°5.** The category $\mathsf{DCube}^{\boxtimes}_M(n)$ has:

- As objects lists of the form $W = (\mathbf{i}_1 : (\!|k_1|\!), \ldots, \mathbf{i}_m : (\!|k_m|\!))$ where the $\mathbf{i}_i$ are regarded as bound de Bruijn indices and the $k_i$ are in $\{0, \ldots, n-1\}$. We define its erasure as $\lfloor W \rfloor = (\mathbf{i}_1 : \mathbb{I}, \ldots, \mathbf{i}_m : \mathbb{I})$.
- As morphisms $\hat{\varphi} : V \to W$, morphisms $\varphi : \lfloor V \rfloor \to \lfloor W \rfloor$ such that for each $\mathbf{i} : (\!|k|\!)$ in $W$, the expression $\mathbf{i}\langle\varphi\rangle$ mentions only variables $\mathbf{j} : (\!|\ell|\!)$ in $V$ such that $\ell \geq k$.

Clearly this category comes with a faithful functor $\lfloor \sqcup \rfloor : \mathsf{DCube}^{\boxtimes}_M(n) \to \mathsf{Cube}^{\boxtimes}_M$.

The categories $\mathsf{DCube}^{\boxslash}_{\mathsf{Pt}_2}(n)$ appear in the model of Degrees of Relatedness [ND18, Nuy18].

**Proposition 1.4.5°6.** The category $\mathsf{DCube}^{\square}_{\mathsf{IPt}_2}(n)$ is isomorphic to the category $\mathsf{JetCube}^{\boxtimes}_{\mathsf{IPt}_2}(\mathsf{f}, [\bigcirc]^n)$ for $\boxtimes \in \{\boxslash, \square\}$.

*Proof.* By theorem 1.4.2°15, the value of $\boxtimes$ does not matter, so let us set $\boxtimes = \square$. We construct a functor $F : \mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{f}, [\bigcirc]^n) \to \mathsf{DCube}^{\square}_{\mathsf{IPt}_2}(n)$ such that $\lfloor \sqcup \rfloor \circ F = \lfloor \sqcup \rfloor$:

- $F(\mathbf{i}_1 : (\!|\frown_{k_1}|\!), \ldots, \mathbf{i}_m : (\!|\frown_{k_m}|\!)) = (\mathbf{i}_1 : (\!|k_1|\!), \ldots, \mathbf{i}_m : (\!|k_m|\!))$,
- For the action on morphisms, we have nothing to choose, we can only verify that it exists. This is done by induction on the derivation in the calculus (fig. 1.1).

It is clear that $F$ is bijective on objects, and faithful. Fullness is proven by proving by induction on the length of the codomain that every morphism of depth $n$ cubes can be derived in the calculus. □

**Proposition 1.4.5°7.** The category $\mathsf{DCube}^{\square}_{\mathsf{Pt}_2}(n)$ is isomorphic to the category $\mathsf{JetCube}^{\boxtimes}_{\mathsf{Pt}_2}(\mathsf{f}, [\bigcirc]^n)$ for $\boxtimes \in \{\boxslash, \square\}$.

*Proof.* Each time, the category for $\mathsf{Pt}_2$ is the wide subcategory of the corresponding one for $\mathsf{IPt}_2$ on morphisms that do not mention $\neg$, so the result follows from proposition 1.4.5°6. □

### 1.4.5 (f) Comparison to Pinyo and Kraus's Twisted Cube Category

In this section, we relate jet cubes to Pinyo and Kraus's twisted cubes [PK20] when $\vec{a} = [\nearrow]$. $\mathsf{JetSet}([\nearrow])$ is the category of proof-irrelevant reflexive graphs. Pinyo and Kraus use arbitrary proof-irrelevant graphs, but since $\top$ is reflexive and the twisted prism functor [PK20, def. 4] restricts to reflexive graphs, all twisted cubes are reflexive graphs anyway.

Two twisted cube categories appear (up to isomorphism) in [PK20], and we show that we can recover both.

**Definition 1.4.5°8.** [PK20, def. 25] The category $\mathsf{TwCube}_{\mathsf{graph}}$ has as objects $[\nearrow]$-jet-cubes (i.e. natural numbers) and as morphisms $V \to W$ all jet set morphisms (i.e. graph morphisms) $\mathsf{JEP}(V) \to \mathsf{JEP}(W)$.

**Proposition 1.4.5°9.** $\mathsf{TwCube}_{\mathsf{graph}}$ is isomorphic to $\mathsf{JetCube}^{\boxslash}_{\mathsf{Boo}}(\mathsf{f}, [\nearrow])$.

*Proof.* Clearly, the objects correspond. The morphisms $V \to W$ of $\mathsf{JetCube}^{\boxslash}_{\mathsf{Boo}}(\mathsf{f}, [\nearrow])$ are morphisms $f : \mathsf{JEP}(V) \to \mathsf{JEP}(W)$ such that $Uf : \mathsf{EP}(\lfloor V \rfloor) \to \mathsf{EP}(\lfloor W \rfloor)$ lifts to a morphism of cubes, which it always does by proposition 1.4.1°7. □

We thoroughly rephrase Pinyo and Kraus's ternary twisted cube category:

**Definition 1.4.5°10.** [PK20, def. 34] The category $\mathsf{TwCube}_{\mathsf{tri}}$ has as objects $[\nearrow]$-jet-cubes (i.e. natural numbers) and as morphisms $V \to W$ all jet set morphisms (i.e. graph morphisms) $\mathsf{JEP}(V) \to \mathsf{JEP}(W)$ or, equivalently by the previous proposition, all jet cube morphisms $V \to W$ in $\mathsf{JetCube}^{\boxslash}_{\mathsf{Boo}}(\mathsf{f}, [\nearrow])$, generated by the rules TERMINAL, [SRC:FWD immediately below INV:BCK], TGT:FWD and PRISM:FWD in fig. 1.1.

The shared reader may object that Pinyo and Kraus define the morphisms of $\mathsf{TwCube}_{\mathsf{tri}}$ by *constructing* them inductively, rather than by selecting them inductively as we do above. However:

**Corollary 1.4.5°11.** Any morphism of $\mathsf{TwCube}_{\mathsf{tri}}$ as defined here, has a unique derivation using the given rules. $\square$

**Proposition 1.4.5°12.** $\mathsf{TwCube}_{\mathsf{tri}}$ is isomorphic to $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{f}, [\nearrow])$.

*Proof.* Since the rules mentioned in definition 1.4.5°10 pertain to the calculus of $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{fbe}, [\nearrow])$ and in fact always yield premises/conclusions about $\mathsf{f}$-jet-cubes for conclusions/premises about $\mathsf{f}$-jet-cubes, and since $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{f}, [\nearrow])$ is a faithful subcategory of $\mathsf{JetCube}^{\boxslash}_{\mathsf{Boo}}(\mathsf{f}, [\nearrow])$ by proposition 1.4.1°8, it is clear that the identity-on-objects functor $\mathsf{TwCube}_{\mathsf{tri}} \to \mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{f}, [\nearrow])$ is faithful, i.e. we can think of $\mathsf{TwCube}_{\mathsf{tri}}$ as a wide subcategory of $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{f}, [\nearrow])$.

To show fullness, we note the following facts about the calculus for $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{fbe}, [\nearrow])$:

1. No rule (read bottom-up) introduces equijets in the codomain.
2. In absence of equijets in the codomain, no rule changes the mask (except SYMMETRIZE applied to the terminal codomain, but then apply TERMINAL instead).
   - In particular, SYMMETRIZE is useless.
3. Equijet variables can only be used at strictly stronger (lower) degrees, or at the current degree $i$ if $a_i = \bigcirc$. Since we have only one directed degree, equijet variables cannot be used.
   - Hence WKN can only be used to derive constant morphisms, which can instead be derived by [SRC:FWD after INV:BCK] and TGT:FWD.
   - Hence EXCHANGE is useless.
4. At mask $[\nearrow]$, the rule CONCURSOR cannot be used as there is only one degree.
5. Since INV:FWD and INV:BCK are mutually inverse, they can together be freely inserted everywhere. Hence, we can replace the rule SRC:FWD with [SRC:FWD after INV:BCK].
6. INV:FWD and INV:BCK can be pushed up through any of the remaining rules except PRISM:FWD and PRISM:BCK. Thus, we only involute right before using a variable. All other rules (still read bottom-up) do not turn an $\mathsf{f}$-codomain into an $\mathsf{fbe}$-codomain, i.e. they do not introduce opposite jets. Thus, we can assume the codomain is an $\mathsf{f}$-jet-cube until we encounter $\neg\mathbf{i}$.
7. This means that until we encounter $\neg\mathbf{i}$, we only need the rules TERMINAL, [SRC:FWD after INV:BCK], TGT:FWD and PRISM:FWD. These do not turn an $\mathsf{f}$-codomain into an $\mathsf{fbe}$-codomain. Thus, until we encounter $\neg\mathbf{i}$, we can assume the domain is an $\mathsf{f}$-jet-cube.
8. If both domain and codomain are forward, we cannot encounter $\neg\mathbf{i}$.

This means we can always rewrite a derivation tree in the calculus for $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{fbe}, [\nearrow])$ of a morphism in $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{f}, [\nearrow])$ to use only the prescribed rules. $\square$

# Chapter 2

# Modalities: Polarized Reshuffles

In this chapter, we pin down the Hom-posets of our mode theory. Section 2.1 defines the 2-poset $\mathsf{PoResh}$ of polarized reshuffles, which we would like to use as our mode theory. The objects of this 2-poset will then be the modes, and as promised in the previous chapter, these are exactly anpolarity masks. The intuitions behind the definition of $\mathsf{PoResh}$ are explained at length in section 2.1.2. $\mathsf{PoResh}$ is a polarized version of the category of reshuffles used to define the mode theory of Degrees of Relatedness [ND18, Nuy18] (theorem 2.1.2°25).

In the following sections, we investigate how polarized reshuffles can act on jet sets (section 2.2), jet topes and presheaves over these (section 2.4), and jet cubes and presheaves over these (section 2.5).

For both presheaf categories, we obtain a no-go theorem (theorems 2.4.3°1 and 2.5.3°1) that tells us that we cannot use *all* of $\mathsf{PoResh}$ as our mode theory. For this reason, we decide to continue with just $\mathsf{PoResh}^{\circ\bullet\circ}$, the 2-poset of polarized reshuffles that have both a left and a right adjoint.

## 2.1 The 2-poset of Polarized Reshuffles

### 2.1.1 Prerequisites

**Definition 2.1.1°1.** A **2-category**, **category-enriched category** or **(2, 2)-category** $\mathcal{C}$ consists of:

- A set of objects $\mathrm{Obj}(\mathcal{C})$,
- For each $x, y \in \mathrm{Obj}(\mathcal{C})$, a *category* of morphisms $\mathrm{Hom}(x, y)$,
- For each $x \in \mathrm{Obj}(\mathcal{C})$, an identity morphism $\mathrm{id}_x \in \mathrm{Obj}(\mathrm{Hom}(x, x))$,
- For each $x, y, z \in \mathrm{Obj}(\mathcal{C})$, a *functor* $\circ : \mathrm{Hom}(y, z) \times \mathrm{Hom}(x, y) \to \mathrm{Hom}(x, z)$,

such that $\circ$ is associative and unital with respect to id.

We write $\mathcal{C}^{\mathsf{op}}$ for the category with the same objects and such that $\mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(y, x)$.

We write $\mathcal{C}^{\mathsf{co}}$ for the category with the same objects and such that $\mathrm{Hom}_{\mathcal{C}^{\mathsf{co}}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, y)^{\mathsf{op}}$.

We write $\mathcal{C}^{\mathsf{co;op}} = (\mathcal{C}^{\mathsf{co}})^{\mathsf{op}}$.

**Definition 2.1.1°2.** A **2-poset**, **poset-enriched category** or **Hom-wise gaunt/skeletal (1, 2)-category** $\mathcal{C}$ consists of:

- A set of objects $\mathrm{Obj}(\mathcal{C})$,
- For each $x, y \in \mathrm{Obj}(\mathcal{C})$, a *poset* of morphisms $\mathrm{Hom}(x, y)$,
- For each $x \in \mathrm{Obj}(\mathcal{C})$, an identity morphism $\mathrm{id}_x \in \mathrm{Hom}(x, x)$,
- For each $x, y, z \in \mathrm{Obj}(\mathcal{C})$, a *monotonically increasing* map $\circ : \mathrm{Hom}(y, z) \times \mathrm{Hom}(x, y) \to \mathrm{Hom}(x, z)$,

such that $\circ$ is associative and unital with respect to id.

### 2.1.2 Definitions and Properties

In this section, we will construct a diagram of 2-posets and poset-valued profunctors between them. Viewing poset-valued presheaves $P : \mathcal{C}^{\mathsf{op}} \to \mathsf{Poset}$ as profunctors to the terminal 2-poset $\top$, the diagram we will construct is the following:

$$\begin{tikzcd}
 & \mathsf{Pol} \arrow[dr, "\mathsf{Ori}", "/" marking] \arrow[d, phantom, "\Downarrow"] & \\
\mathsf{PoResh} \arrow[ur, "\mathsf{PolDeg}^{\top}_{\mathrm{eq}}", "/" marking] \arrow[rr, "\mathsf{OriDeg}^{\top}_{\mathrm{eq}}"', "|" marking] & & \top .
\end{tikzcd}$$

The 2-poset $\mathsf{PoResh}$ will serve as the mode theory of Naturality Pretype Theory. Its objects, the modes, will be anpolarity masks. Its morphisms, the modalities, are called **polarized reshuffles** and will answer the question: for a $\mu$-modal function $f$, in order to get a $j$-jet $f(x) \mathrel{-\!\!\triangleright}_j f(y)$, what do I need between $x$ and $y$? The answer $P_i$ consists of two parts: the *degree* $i$ at which we need a jet between $x$ and $y$, and the *orientation* $P$ of that jet. A polarized reshuffle $\mu \in \mathrm{Hom}_{\mathsf{PoResh}}(\vec{a}, \vec{b})$ is merely that: a consistent answer to the above question for all degrees $j$ in the codomain mask $\vec{b}$. However, if we can answer that question, we can more generally answer: in order to get $f(x)\ Q_j\ f(y)$, what do I need between $x$ and $y$? The $Q_j$ will be called an **oriented degree** for the mask $\vec{b}$, i.e. $Q_j \in \mathsf{OriDeg}^{\top}_{\mathrm{eq}}(\vec{b})$, and we can combine it with $\mu$ to get the answer $P_i$ to our question, which is an oriented degree for the domain mask $\vec{a}$, i.e. $P_i \in \mathsf{OriDeg}^{\top}_{\mathrm{eq}}(\vec{a})$. The operation that answers this question, is the functorial action of the presheaf $\mathsf{OriDeg}^{\top}_{\mathrm{eq}}$.

In order to be able to define $\mathsf{PoResh}$ in a somewhat concise way, we start by considering what would happen if there were always only a single degree. This leads to the 2-poset $\mathsf{Pol}$ whose objects are anpolarities and whose morphisms are polarities, which answer the question: in order to get $f(x) \mathrel{-\!\!\triangleright} f(y)$, what do I need between $x$ and $y$? There will be 4 polarities $\circledast$, $\oplus$, $\ominus$ and $\otimes$ which will ask for an equijet, forward jet, backward jet and an infrajet respectively. Once we know the answer to that question, we can answer the more general question: for a $p$-polar function $f$, in order to get $f(x)\ Q\ f(y)$, what do I need between $x$ and $y$? In other words, there will be an operation combining a polarity $p$ with an orientation $Q$, producing another orientation $P$. This is the functorial action of the presheaf of orientations $\mathsf{Ori}$. Finally, a **polarized degree** $p_i$ can be regarded as a modality for a function $f : A \to B$ from a domain $A$ with multiple degrees, to a codomain $B$ with only a single degree. It answers the question: for a $p_i$-modal function $f$, in order to get $f(x) \mathrel{-\!\!\triangleright} f(y)$, what do I need between $x$ and $y$? The answer consists of a degree $i$ at which a jet is needed in $A$, and a polarity telling us which way the jet should point. This multi-single-degree function $f$ can be composed at the domain side with multi-multi-degree functions, and at the codomain side with single-single-degree functions, which explains why $\mathsf{PolDeg}^{\top}_{\mathrm{eq}}$ is a profunctor. Finally, we can generalize the above question: for a $p_i$-modal function $f$, in order to get $f(x)\ Q\ f(y)$, what do I need between $x$ and $y$? Thus, we get an operation that takes an orientation $Q$ and a polarized degree $p_i$ and answers with a polarized degree $P_i$; this is the filler of the triangular diagram. Naturality and extranaturality of this operation simply express that it is consistent with how multi-single-degree functions compose on either side.

For functions with multiple degrees in the codomain, we know that $f(x) \mathrel{-\!\!\triangleright}_i f(y)$ will imply $f(x) \mathrel{\triangleleft\!\!\triangleright}_{i+1} f(y)$. Hence, for a modality $\mu$ to be consistent, we will require that $\mathrel{-\!\!\triangleright}_i \circ\, \mu \leq \mathrel{\triangleleft\!\!\triangleright}_{i+1} \circ\, \mu$. Since the only informational content of $\mu$ is parametrized by a *degree* $i$, we will require that $i \cdot \mu \ll (i+1) \cdot \mu$. This is the motivation of the somewhat peculiar relation $\ll$ introduced below, which is always transitive, reflexive on symmetric objects and antireflexive on non-symmetric objects.

#### 2.1.2 (a) Polarities and Orientations

**Definition 2.1.2°1.** The **2-poset of polarities** $\mathsf{Pol}$ has

- as objects, anpolarities, i.e. $\bigcirc$ and $\nearrow$,

- as morphisms, **polarities**:

$$\begin{aligned}\mathrm{Hom}(\nearrow, \nearrow) &= \{\otimes, \oplus, \ominus, \circledast\},\\ \mathrm{Hom}(\nearrow, \bigcirc) &= \{\otimes, \circledast\},\\ \mathrm{Hom}(\bigcirc, \nearrow) &= \{\otimes = \oplus = \ominus = \circledast =: \odot\},\\ \mathrm{Hom}(\bigcirc, \bigcirc) &= \{\otimes = \oplus = \ominus = \circledast =: \odot\}.\end{aligned}$$

  In other words, all Hom-sets are a quotient of a subset of $\mathrm{Hom}(\nearrow, \nearrow)$. If the domain is $\bigcirc$, we identify all polarities as $\odot$. If the domain is $\nearrow$ but the codomain is $\bigcirc$, we remove $\oplus$ and $\ominus$.
  The order is always generated by the fact that $\otimes$ is minimal and $\circledast$ is maximal. The identities are $\mathrm{id}_{\nearrow} = \oplus$ and $\mathrm{id}_{\bigcirc} = \odot$ and composition is defined in fig. 2.1 and is monotonically increasing in both arguments.
  We call these polarities: $\otimes$ mixed-polar, $\oplus$ positive, $\ominus$ negative, $\circledast$ isopolar and $\odot$ non-polar.

**Definition 2.1.2°2.** The poset-valued **presheaf of orientations** $\mathsf{Ori} : \mathsf{Pol}^{\mathsf{op}} \to \mathsf{Poset}$ has, as its action on objects:

$$\mathsf{Ori}(a) = \{\leftrightarrow, \rightarrow, \leftarrow, \rightleftarrows\},$$

where we identify all elements with $\frown$ if $a = \bigcirc$. The order is generated by the fact that $\leftrightarrow$ is minimal and $\rightleftarrows$ is maximal. The action on morphisms $\mathsf{Ori}(p)$ is denoted as $\sqcup \bullet p$ and given in fig. 2.1; it is monotonically increasing in both arguments.[1]

| $\downarrow \circ \rightarrow$ | $\otimes$ | $\oplus$ | $\ominus$ | $\circledast$ |
|---|---|---|---|---|
| $\otimes$ | $\otimes$ | $\otimes$ | $\otimes$ | $\circledast$ |
| $\oplus$ | $\otimes$ | $\oplus$ | $\ominus$ | $\circledast$ |
| $\ominus$ | $\otimes$ | $\ominus$ | $\oplus$ | $\circledast$ |
| $\circledast$ | $\otimes$ | $\circledast$ | $\circledast$ | $\circledast$ |

| $\downarrow \bullet \rightarrow$ | $\otimes$ | $\oplus$ | $\ominus$ | $\circledast$ |
|---|---|---|---|---|
| $\leftrightarrow$ | $\leftrightarrow$ | $\leftrightarrow$ | $\leftrightarrow$ | $\rightleftarrows$ |
| $\rightarrow$ | $\leftrightarrow$ | $\rightarrow$ | $\leftarrow$ | $\rightleftarrows$ |
| $\leftarrow$ | $\leftrightarrow$ | $\leftarrow$ | $\rightarrow$ | $\rightleftarrows$ |
| $\rightleftarrows$ | $\leftrightarrow$ | $\rightleftarrows$ | $\rightleftarrows$ | $\rightleftarrows$ |

Figure 2.1: Composition of polarities and their action on orientations.

**Proposition 2.1.2°3.** The operations in fig. 2.1 are well-defined.

*Proof.* We prove this for composition in $\mathsf{Pol}$; the proof for $\mathsf{Ori}$ is analogous. Consider a composition $a \xrightarrow{p} b \xrightarrow{q} c$. Special situations may occur if either $a$, $b$ or $c$ equals $\bigcirc$.

- If $a = \bigcirc$, then $p = \odot$ we need to make sense of $q \circ \odot$, which also equals $\odot$.
- If $b = \bigcirc$, then $q = \odot$ and $p \in \{\otimes, \circledast\}$ and we need to make sense of $\odot \circ p$, which according to fig. 2.1 equals $p$.
- If $c = \bigcirc$, then $q \in \{\otimes, \circledast\}$ and the same needs to hold for $q \circ p$, which is indeed the case. $\square$

**Proposition 2.1.2°4.** Composition of polarities is unital and associative. In fact, composition of a sequence of $n \geq 0$ polarities can be characterized as follows:

- If any of the operands is either $\otimes$ or $\circledast$, then the result is $\otimes$ or $\circledast$, whichever occurs most to the right.
- If all of the operands are either $\oplus$ or $\ominus$, then the result is $\ominus$ if $\ominus$ occurs as an operand an odd number of times, and $\oplus$ otherwise. $\square$

**Proposition 2.1.2°5.** The presheaf of orientations $\mathsf{Ori}$ is isomorphic to the Yoneda-embedding $\mathbf{y}(\nearrow) = \mathrm{Hom}(\sqcup, \nearrow)$ of $\nearrow$ in two ways: by identifying $\oplus$ with either $\rightarrow$ or $\leftarrow$.

In particular, it satisfies the presheaf laws: it respects identity and composition of polarities. $\square$

[1] So the domain of $\mathsf{Ori} : \mathsf{Pol}^{\mathsf{op}} \to \mathsf{Poset}$ has reversed 1-cells but not reversed 2-cells / inequalities.

**Proposition 2.1.2°6.** Each constituent poset $\mathrm{Hom}(a, b)$ of $\mathsf{Pol}$ and each constituent poset $\mathsf{Ori}(a)$ of $\mathsf{Ori}$ is a lattice. Composition of polarities and their action on orientations both preserve meets and joins in the *left* argument. $\square$

Note that the fact that $\sqcup \circ \sqcup$ and $\sqcup \bullet \sqcup$ preserve meets and joins in their left argument, is easy to explain from the intuitions in section 2.1.2. Indeed, $(P \vee Q) \bullet p$ answers the question: for a $p$-polar single-single degree function $f$, in order to have either $f(x)\ P\ f(y)$ or $f(x)\ Q\ f(y)$, what do I need between $x$ and $y$? The answer is: either the weakest precondition $P \bullet p$ of $f(x)\ P\ f(y)$, or the weakest precondition $Q \bullet p$ of $f(x)\ Q\ f(y)$, i.e. you need $(P \bullet p) \vee (Q \bullet p)$. The argument for meets and for $\sqcup \circ \sqcup$ is similar.

**Definition 2.1.2°7.** We call a polarity $p : a \to b$ **symmetric** if $\leftarrowtriangle \bullet p = \rightarrowtriangle \bullet p$, which is the case if $a = \bigcirc$ or $b = \bigcirc$ or $p \in \{\circledast, \otimes\}$.

For $p, q : a \to b$, we write $p \ll q$ if one of the following, equivalent conditions is satisfied:

1. $\rightarrowtriangle \bullet p \leq \rightarrowtriangle \bullet q$ and $\rightarrowtriangle \bullet p \leq \leftarrowtriangle \bullet q$,
2. $\rightarrowtriangle \bullet p \leq \rightarrowtriangle \bullet q$ and $\leftarrowtriangle \bullet p \leq \rightarrowtriangle \bullet q$,
3. $\rightarrowtriangle \bullet p \leq \leftrightarrowtriangle \bullet q$,
4. $\rightleftarrows \bullet p \leq \rightarrowtriangle \bullet q$,
5. $\rightleftarrows \bullet p \leq \leftrightarrowtriangle \bullet q$,
6. $p \leq q$ and at least either of them is symmetric.

*Proof of equivalence.* $1 \Leftrightarrow 2$ This is easy to check manually.
$1 \Leftrightarrow 3$ By the universal property of the meet, and the fact that $\sqcup \bullet q$ preserves meets.
$2 \Leftrightarrow 4$ By the universal property of the join, and the fact that $\sqcup \bullet p$ preserves joins.
$1 \wedge 2 \Leftrightarrow 5$ By similar reasoning as the two previous points.
$1 \Rightarrow 6$ Condition 1 implies $p \leq q$ because $\rightarrowtriangle \bullet \sqcup$ is an isomorphism of posets. If $p \leq q$ but neither is symmetric, then $p = q \in \{\oplus \neq \ominus\}$, contradicting 1.
$6 \Rightarrow 1 \vee 2$ We can immediately prove either 1 or 2 depending on whether $p$ or $q$ is symmetric. $\square$

**Definition 2.1.2°8.** We call an orientation $P \in \mathsf{Ori}(a)$ **symmetric** if $a = \bigcirc$ or $P \in \{\leftrightarrowtriangle, \rightleftarrows\}$.

For $P, Q \in \mathsf{Ori}(a)$, we write $P \ll Q$ if $P \leq Q$ and at least either of them is symmetric.

#### 2.1.2 (b) Polarized Degrees

**Definition 2.1.2°9.** Given an anpolarity mask $\vec{a}$, we define the following posets of **degrees**:

- $\mathsf{Deg}(\vec{a}) = \{0 \leq 1 \leq \ldots \leq \mathsf{len}(\vec{a}) - 1\}$,
- $\mathsf{Deg}_{\mathrm{eq}}(\vec{a}) = \{\mathrm{eq} \leq 0 \leq 1 \leq \ldots \leq \mathsf{len}(\vec{a}) - 1\}$,
- $\mathsf{Deg}^{\top}_{\mathrm{eq}}(\vec{a}) = \{\mathrm{eq} \leq 0 \leq 1 \leq \ldots \leq \mathsf{len}(\vec{a}) - 1 \leq \top\}$.

We generalize the notation $a_i$ to generalized degrees $i$ by setting $a_{\mathrm{eq}} = a_{\top} = \bigcirc$.

**Definition 2.1.2°10.** Given an anpolarity mask $\vec{a}$ and an anpolarity $b$, we define the following posets of **polarized degrees**:

- $\mathsf{PolDeg}(\vec{a}, b) = \Sigma(i \in \mathsf{Deg}(\vec{a})).\mathrm{Hom}_{\mathsf{Pol}}(a_i, b)$ with dictionary order,
- $\mathsf{PolDeg}_{\mathrm{eq}}(\vec{a}, b)$ and $\mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, b)$ similarly.

We denote a dependent pair $(i, p)$ of a degree $i$ and a polarity $p$ as $p_i$ and write $\mathsf{deg}(p_i) = i$ and $\mathsf{pol}(p_i) = p$.

We write $p_i \ll q_j$ if **EITHER** $i < j$ **OR** $i = j$ and $p \ll q$.

We make $\mathsf{PolDeg}(\vec{a}, b)$ a covariant 2-functor of $b \in \mathrm{Obj}(\mathsf{Pol})$ by defining $q \circ (p_i) = (q \circ p)_i$, and similar for $\mathsf{PolDeg}_{\mathrm{eq}}(\vec{a}, b)$ and $\mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, b)$.

**Corollary 2.1.2°11.** The functor $\mathsf{PolDeg}(\vec{a}, \sqcup)$ (as well as $\mathsf{PolDeg}_{\mathrm{eq}}(\vec{a}, \sqcup)$ and $\mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, \sqcup)$) respects identity and composition of polarities.

*Proof.* This is immediately inherited from associativity and unitality of $\mathsf{Pol}$. $\square$

**Corollary 2.1.2°12.** The functorial action $q \circ (p_i)$ of polarities on polarized degrees, preserves meets and joins in its left argument. $\square$

**Proposition 2.1.2°13.** Each poset $\mathsf{PolDeg}(\vec{a}, b)$, $\mathsf{PolDeg}_{\mathrm{eq}}(\vec{a}, b)$ and $\mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, b)$ is a lattice.

*Proof.* To take the meet/join of polarized degrees with different degrees, simply take the operand with the lowest/highest degree. To take the meet/join of polarized degrees with the same degree, take the meet/join of the polarities. $\square$

### 2.1.2 (c) Polarized Reshuffles

**Definition 2.1.2°14.** We define the 2-poset $\mathsf{PoResh}$ of **polarized reshuffles** as follows:

- Objects are anpolarity masks.
- Morphisms $\mu : \vec{a} \to \vec{b}$ are functions

  $$\sqcup \cdot \mu : (j : \mathsf{Deg}_{\mathrm{eq}}(\vec{b})) \to \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, b_j)$$

  such that if $j < j'$, then $j \cdot \mu \ll j' \cdot \mu$.
  - We extend the domain of $\sqcup \cdot \mu$ to $\mathsf{Deg}^{\top}_{\mathrm{eq}}(\vec{b})$ by setting $\top \cdot \mu = \odot_{\top}$.
    This maintains the property that $i < j$ implies $i \cdot \mu \ll j \cdot \mu$.
  - Given an anpolarity $c$, we lift $\sqcup \cdot \mu$ to a function

    $$\sqcup \circ \mu : (q_j : \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{b}, c)) \to \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, c) : q_j \mapsto q \circ (j \cdot \mu). \tag{2.1}$$

    This has the property that if $q_j \ll {q'}_{j'}$, then $q_j \circ \mu \ll {q'}_{j'} \circ \mu$.
  - The identity $\mathrm{id} : \vec{a} \to \vec{a}$ is given by $i \cdot \mathrm{id} := \oplus_i$.
  - Composition $\nu \circ \mu$ is given by $i \cdot (\nu \circ \mu) := (i \cdot \nu) \circ \mu$.
- Inequality of morphisms is considered pointwise, i.e. $\mu \leq \nu$ if for all $j$, we have $j \cdot \mu \leq j \cdot \nu$.

We will also denote modalities $\mu : \vec{a} \to \vec{b}$ as $\mu = \langle \mathrm{eq} \cdot \mu \,|\, 0 \cdot \mu, \ldots, (\mathsf{len}(\vec{a}) - 1) \cdot \mu \rangle$, e.g. $\mathrm{id}_{[\not\nearrow, \bigcirc, \not\nearrow]} = \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \odot_1, \oplus_2 \rangle$.

**Corollary 2.1.2°15.** We have $\oplus_j \circ \mu = j \cdot \mu$. $\square$

**Proposition 2.1.2°16.** $\mathsf{PolDeg}^{\top}_{\mathrm{eq}}$ is a profunctor from $\mathsf{PoResh}$ to $\mathsf{Pol}$, i.e. a 2-functor $\mathsf{PolDeg}^{\top}_{\mathrm{eq}} : \mathsf{PoResh}^{\mathsf{op}} \times \mathsf{Pol} \to \mathsf{Poset}$.

We do not have the same result for $\mathsf{PolDeg}$, and do not bother for $\mathsf{PolDeg}_{\mathrm{eq}}$.

*Proof.* We already knew that it was covariant in its second argument. Contravariance in the first argument is given by the action $\sqcup \circ \mu$ on polarized degrees. To see that it respects the identity:

$$p_i \circ \mathrm{id} = p \circ (i \cdot \mathrm{id}) = p \circ (\oplus_i) = (p \circ \oplus)_i = p_i.$$

To see that functoriality on both sides commutes:

$$\begin{aligned}
(r \circ q_j) \circ \mu &= (r \circ q)_j \circ \mu && \text{(definition 2.1.2°10)} \\
&= (r \circ q) \circ (j \cdot \mu) && \text{(eq. (2.1))} \\
&= r \circ (q \circ (j \cdot \mu)) && (\mathsf{PolDeg}^{\top}_{\mathrm{eq}} \text{ respects comp. of polarities}) \\
&= r \circ (q_j \circ \mu) && \text{(eq. (2.1))}.
\end{aligned}$$

For composition:

$$\begin{aligned}
r_k \circ (\nu \circ \mu) &= r \circ (k \cdot (\nu \circ \mu)) && \text{(eq. (2.1))} \\
&= r \circ ((k \cdot \nu) \circ \mu) && (\text{def. of comp. in } \mathsf{PoResh}) \\
&= (r \circ (k \cdot \nu)) \circ \mu && (\text{functoriality of } \mathsf{PolDeg}^{\top}_{\mathrm{eq}} \text{ on both sides commutes}) \\
&= (r_k \circ \nu) \circ \mu && \text{(eq. (2.1))}.
\end{aligned}$$

$\square$

**Proposition 2.1.2°17.** Composition in PoResh is unital and associative.

*Proof.* For the right unit law, we have:

$$\begin{aligned} i \cdot (\mu \circ \mathrm{id}) &= (i \cdot \mu) \circ \mathrm{id} && \text{(def. of comp. in PoResh)} \\ &= i \cdot \mu && (\mathsf{PolDeg}^{\top}_{\mathrm{eq}} \text{ respects the identity}). \end{aligned}$$

For associativity, we have:

$$\begin{aligned} i \cdot (\rho \circ (\nu \circ \mu)) &= (i \cdot \rho) \circ (\nu \circ \mu) && \text{(def. of comp. in PoResh)} \\ &= ((i \cdot \rho) \circ \nu) \circ \mu && (\mathsf{PolDeg}^{\top}_{\mathrm{eq}} \text{ respects composition}) \\ &= (i \cdot (\rho \circ \nu)) \circ \mu && \text{(def. of comp. in PoResh)} \\ &= i \cdot ((\rho \circ \nu) \circ \mu) && \text{(def. of comp. in PoResh).} \end{aligned}$$

For the left unit law, we have

$$\begin{aligned} i \cdot (\mathrm{id} \circ \mu) &= (i \cdot \mathrm{id}) \circ \mu && \text{(def. of comp. in PoResh)} \\ &= \oplus_i \circ \mu && \text{(def. of id in PoResh)} \\ &= i \cdot \mu && \text{(corollary 2.1.2°15).} \end{aligned}$$

☐

**Proposition 2.1.2°18.**
1. Each Hom-set of PoResh is a lattice.
2. The action $q_j \circ \mu$ of polarized degrees on polarized reshuffles, preserves meets and joins in its left argument.
3. Composition of polarized reshuffles, preserves meets and joins in its left argument.

*Proof.*
1. We can take meets/joins pointwise.
2. For polarized degrees with different degrees, meets/joins amount to minima/maxima, which are preserved by any poset morphism. For polarized degrees with the same degree, this follows, via eq. (2.1), from 2.1.2°12.
3. Follows immediately from the previous two points and the definition of composition of polarized reshuffles. ☐

### 2.1.2 (d) Oriented Degrees

**Definition 2.1.2°19.** Given an anpolarity mask $\vec{a}$, we define the following posets of **oriented degrees**:

- $\mathsf{OriDeg}(\vec{a}) = \Sigma(i \in \mathsf{Deg}(\vec{a})).\mathsf{Ori}(a_i)$ with dictionary order,
- $\mathsf{OriDeg}_{\mathrm{eq}}(\vec{a})$ and $\mathsf{OriDeg}^{\top}_{\mathrm{eq}}(\vec{a})$ similarly.

We denote a dependent pair $(i, P)$ of a degree $i$ and an orientation $P$ as $P_i$ and write $\mathsf{deg}(P_i) = i$ and $\mathsf{ori}(P_i) = P$.

We write $P_i \ll Q_j$ if **EITHER** $i < j$ **OR** $i = j$ and $P \ll Q$.

We define a transformation, extranatural in $b$ (proposition 2.1.2°20):

$$\sqcup \bullet \sqcup : \mathsf{Ori}(b) \times \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, b) \to \mathsf{OriDeg}^{\top}_{\mathrm{eq}}(\vec{a}) : Q \bullet (p_i) = (Q \bullet p)_i, \tag{2.2}$$

and similar operations for $\mathsf{PolDeg}_{\mathrm{eq}}$ and $\mathsf{PolDeg}$.

Using this, we make $\mathsf{OriDeg}^{\top}_{\mathrm{eq}}$ (but not $\mathsf{OriDeg}$ and $\mathsf{OriDeg}_{\mathrm{eq}}$) a presheaf by defining an operation

$$\sqcup \bullet \sqcup : \mathsf{OriDeg}^{\top}_{\mathrm{eq}}(\vec{b}) \times \mathrm{Hom}_{\mathsf{PoResh}}(\vec{a}, \vec{b}) \to \mathsf{OriDeg}^{\top}_{\mathrm{eq}}(\vec{a}) : Q_j \bullet \mu = Q \bullet (j \cdot \mu), \tag{2.3}$$

which preserves identity and composition (proposition 2.1.2°22). We can now observe that the action in eq. (2.2) is natural in $\vec{a}$ (proposition 2.1.2°21).

**Proposition 2.1.2°20.** The action in eq. (2.2) is extranatural in $b$.

*Proof.* This is immediately inherited from functoriality of $\mathsf{Ori}$:

$$\begin{aligned}
(R \bullet q) \bullet (p_i) &= ((R \bullet q) \bullet p)_i && \text{(eq. (2.2))} \\
&= (R \bullet (q \circ p))_i && \text{(Ori respects comp. of polarities)} \\
&= R \bullet ((q \circ p)_i) && \text{(eq. (2.2))} \\
&= R \bullet (q \circ (p_i)) && \text{(definition 2.1.2°10).} \qquad \square
\end{aligned}$$

**Proposition 2.1.2°21.** The action in eq. (2.2) is natural in $\vec{a}$.

*Proof.*

$$\begin{aligned}
(R \bullet q_j) \bullet \mu &= (R \bullet q)_j \bullet \mu && \text{(eq. (2.2))} \\
&= (R \bullet q) \bullet (j \cdot \mu) && \text{(eq. (2.3))} \\
&= R \bullet (q \circ (j \cdot \mu)) && \text{(proposition 2.1.2°20)} \\
&= R \bullet (q_j \circ \mu) && \text{(eq. (2.1)).} \qquad \square
\end{aligned}$$

**Proposition 2.1.2°22.** The functorial action of $\mathsf{OriDeg}^{\top}_{\mathrm{eq}}$ respects identity and composition of polarized reshuffles.

*Proof.* For the identity, we have

$$\begin{aligned}
P_i \bullet \mathrm{id} &= P \bullet (i \cdot \mathrm{id}) && \text{(eq. (2.3))} \\
&= P \bullet (\oplus_i) && \text{(def. of id in PoResh)} \\
&= (P \bullet \oplus)_i && \text{(eq. (2.2))} \\
&= P_i && \text{(Ori respects identity polarity).}
\end{aligned}$$

For composition, we have

$$\begin{aligned}
R_k \bullet (\nu \circ \mu) &= R \bullet (k \cdot (\nu \circ \mu)) && \text{(eq. (2.3))} \\
&= R \bullet ((k \cdot \nu) \circ \mu) && \text{(def. of comp. in PoResh)} \\
&= (R \bullet (k \cdot \nu)) \bullet \mu && \text{(proposition 2.1.2°21)} \\
&= (R_k \bullet \nu) \bullet \mu && \text{(eq. (2.3)).} \qquad \square
\end{aligned}$$

**Proposition 2.1.2°23.**
1. Each poset $\mathsf{OriDeg}(\vec{a})$, $\mathsf{OriDeg}_{\mathrm{eq}}(\vec{a})$ and $\mathsf{OriDeg}^{\top}_{\mathrm{eq}}(\vec{a})$ is a lattice.
2. The action in eq. (2.2) preserves meets and joins in its left argument.
3. The action in eq. (2.3) preserves meets and joins in its left argument.

*Proof.*
1. To take the meet/join of oriented degrees with different degrees, simply take the operand with the lowest/highest degree. To take the meet/join of oriented degrees with the same degree, take the meet/join of the orientations.
2. Trivial from proposition 2.1.2°6.
3. For oriented degrees with different degrees, meets/joins amount to minima/maxima, which are preserved by any poset morphism. For oriented degrees with the same degree, this follows, via eq. (2.3), from the previous point. □

#### 2.1.2 (e) Discussion and Related Work

**When Everything is Symmetric** If we forget about the existence of the anpolarity $\nearrow\!\!\!\!\times$, then $\mathsf{Pol}$ becomes a 2-poset with only a single object $\bigcirc$ and a single morphism $\odot \in \mathrm{Hom}_{\mathsf{Pol}}(\bigcirc, \bigcirc)$. The presheaf $\mathsf{Ori}$ is then just a poset $\mathsf{Ori}(\bigcirc)$ containing only a single orientation $\frown$. An anpolarity mask $\vec{a}$ is always of the form $[\bigcirc]^n$, i.e. it is just a natural number. Furthermore, we have

$$\mathsf{PolDeg}^{\top}_{\mathrm{eq}}([\bigcirc]^n, \bigcirc) \cong \mathsf{OriDeg}^{\top}_{\mathrm{eq}}([\bigcirc]^n) \cong \mathsf{Deg}^{\top}_{\mathrm{eq}}([\bigcirc]^n).$$

Finally, a polarized reshuffle $\mu \in \mathrm{Hom}_{\mathsf{PoResh}}([\bigcirc]^m, [\bigcirc]^n)$ is essentially just a function

$$\sqcup \cdot \mu : \mathsf{Deg}_{\mathrm{eq}}([\bigcirc]^n) \to \mathsf{Deg}^{\top}_{\mathrm{eq}}([\bigcirc]^m).$$

Thus, we have retrieved exactly the 2-poset of (ordinary) reshuffles from Degrees of Relatedness [ND18, Nuy18]:

**Definition 2.1.2°24** ([Nuy18, def. 6.2.7])**.** The **2-poset of reshuffles** $\mathsf{Resh}$ has:

- as objects, **depths**, which are integers $d \geq -1$,
- as morphisms $\mu \in \mathrm{Hom}_{\mathsf{Resh}}(d, e)$, functions

$$\sqcup \cdot \mu : \{\mathrm{eq} \leq 0 \leq \ldots \leq e\} \to \{\mathrm{eq} \leq 0 \leq \ldots \leq d \leq \top\},$$

- pointwise inequalities.

**Theorem 2.1.2°25.** There is an embedding $I : \mathsf{Resh} \to \mathsf{PoResh}$ of the 2-poset of reshuffles $\mathsf{Resh}$ from [Nuy18, def. 6.2.7] into $\mathsf{PoResh}$ which

- sends the depth $d \geq -1$ to the anpolarity mask $I(d) = [\bigcirc]^{d+1}$,
- sends the reshuffle $\mu \in \mathrm{Hom}_{\mathsf{Resh}}(d, e)$ to the polarized reshuffle $I(\mu) \in \mathrm{Hom}_{\mathsf{PoResh}}([\bigcirc]^{d+1}, [\bigcirc]^{e+1})$ such that $i \cdot I(\mu) = \odot_{i \cdot \mu}$.

The action of the 2-functor $I$ on every Hom-poset is an isomorphism of posets. $\square$

**When Everything is Directed** If we forget about the existence of the anpolarity $\bigcirc$, then $\mathsf{Pol}$ becomes a 2-poset with a single object $\nearrow$, i.e. fully determined by the ordered monoid $\mathrm{Hom}_{\mathsf{Pol}}(\nearrow, \nearrow) = \{\circledast, \oplus, \ominus, \otimes\}$.

This ordered monoid has a semantics in endofunctors on $\mathsf{Cat}$, sending $\oplus$ to the identity functor, $\ominus$ to taking the opposite, $\circledast$ to taking the localization (which makes all morphisms invertible) and $\otimes$ to taking the core (which discards all non-invertible morphisms).

*Almost* the same ordered monoid is used by Abel [Abe08] (who writes $\top$ for $\circledast$ and $\circ$ for $\otimes$), in my master thesis [Nuy20a] (where I write $=$ for $\circledast$) and in our earlier work on polarities for Agda [PEC+23] (ignoring the strictly positive modality; there we write $*$ or `mixed` for $\otimes$ and `unused` for $\circledast$). However, there is a subtle difference in that all of the cited work has $\circledast \circ \otimes = \circledast$, whereas we have $\circledast \circ \otimes = \otimes$. The variation with $\circledast \circ \otimes = \circledast$ has a semantics in endofunctors on $\mathsf{Cat}$ where we model $\circledast$ not as taking the localization, but as squashing to a (sub)singleton. Alternatively, it has a semantics on the category of pro-arrow equipments [nLaa, Woo82, Woo85] where $\oplus$ is still the identity, $\ominus$ and $\otimes$ take the opposite and core (resp.) of the arrow dimension, and $\circledast$ replaces the arrow dimension with the core of the pro-arrow dimension.

For us, pro-arrows will live at a different degree than arrows, and a modality that replaces the arrow relation with the core of the pro-arrow relation, will make crucial use of the *reshuffling* aspect of polarized reshuffles. Similarly, a squashing operation can be represented by a modality $\mu$ such that $\mathrm{eq} \cdot \mu = \odot_{\top}$. Our *polarities*, on the contrary, are concerned with only a single degree, and as such we stick to the interpretation of $\circledast$ as promoting infrajets to jets (hence equijets, by symmetry of infrajets), much like the localization does. This leads to the equality $\circledast \circ \otimes = \otimes$.

The presheaf of orientations $\mathsf{Ori}$ is just a single poset $\mathsf{Ori}(\nearrow) = \{\leftrightarrow, \rightarrow, \leftarrow, \rightleftarrows\}$ on which the ordered monoid of polarities has a right action.

An anpolarity mask $\vec{a}$ is always of the form $[\nearrow]^n$, i.e. it is just a natural number. A polarized/oriented degree $p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}([\nearrow]^n, \nearrow)$ or $P_i \in \mathsf{OriDeg}^{\top}_{\mathrm{eq}}([\nearrow]^n)$ is now just a (non-dependent) pair of a degree and a polarity/orientation.

For purposes of illustration, we list all polarized reshuffles in $\mathrm{Hom}_{\mathsf{PoResh}}([\nearrow], [\nearrow])$:

$$\begin{array}{llll}
\langle \odot_{\mathrm{eq}} \mid \odot_{\mathrm{eq}} \rangle & & & \\
\langle \odot_{\mathrm{eq}} \mid \otimes_0 \rangle & \langle \otimes_0 \mid \otimes_0 \rangle & & \\
\langle \odot_{\mathrm{eq}} \mid \oplus_0 \rangle & \langle \otimes_0 \mid \oplus_0 \rangle & & \\
\langle \odot_{\mathrm{eq}} \mid \ominus_0 \rangle & \langle \otimes_0 \mid \ominus_0 \rangle & & \\
\langle \odot_{\mathrm{eq}} \mid \circledast_0 \rangle & \langle \otimes_0 \mid \circledast_0 \rangle & \langle \circledast_0 \mid \circledast_0 \rangle & \\
\langle \odot_{\mathrm{eq}} \mid \odot_{\top} \rangle & \langle \otimes_0 \mid \odot_{\top} \rangle & \langle \circledast_0 \mid \odot_{\top} \rangle & \langle \odot_{\top} \mid \odot_{\top} \rangle,
\end{array}$$

where the identity is $\mathrm{id}_{[\nearrow]} = \langle \odot_{\mathrm{eq}} \mid \oplus_0 \rangle$.

### 2.1.3 Adjunctions of Polarized Reshuffles

**Notation 2.1.3°1.** Sometimes, we need to consider only reshuffles that have at least a certain number of left/right adjoints. The 2-posets that contain only those reshuffles (with the same order relation) will be denoted as $\mathsf{PoResh}^{\circ\circ\bullet\circ}$, where the number of white bullets on the left/right is the minimal number of left/right adjoints.

#### 2.1.3 (a) Characterization

**Theorem 2.1.3°2.** 1. A pair of polarized reshuffles $\lambda \in \mathrm{Hom}(\vec{a}, \vec{b})$ and $\rho \in \mathrm{Hom}(\vec{b}, \vec{a})$ is adjoint ($\lambda \dashv \rho$) if and only if the following equivalent conditions are satisfied:

$$\forall P_i \in \mathsf{OriDeg}^{\top}_{\mathrm{eq}}(\vec{a}), Q_j \in \mathsf{OriDeg}^{\top}_{\mathrm{eq}}(\vec{b}).P_i \leq Q_j \bullet \lambda \Leftrightarrow P_i \bullet \rho \leq Q_j, \tag{2.4}$$

$$\forall c \in \mathbb{A}, p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, c), q_j \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{b}, c).p_i \leq q_j \circ \lambda \Leftrightarrow p_i \circ \rho \leq q_j. \tag{2.5}$$

2. By consequence:
   - A reshuffle $\rho$ has a left adjoint $\lambda$ if and only if

     $$\mathrm{eq} \cdot \rho = \odot_{\mathrm{eq}}, \text{ and} \tag{2.6}$$

     $$\text{for all } i \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a}) \text{ with } i \cdot \rho = \circledast_j, \text{ we have } b_j = \bigcirc \text{ (so } i \cdot \rho = \odot_j). \tag{2.7}$$

     In this case,
     - for any $Q_j \in \mathsf{OriDeg}^{\top}_{\mathrm{eq}}(\vec{b})$, we find $Q_j \bullet \lambda$ as the greatest oriented degree $P_i \in \mathsf{OriDeg}^{\top}_{\mathrm{eq}}(\vec{a})$ such that $P_i \bullet \rho \leq Q_j$;
     - for any $q_j \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{b}, c)$, we find $q_j \circ \lambda$ as the greatest polarized degree $p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, c)$ such that $p_i \circ \rho \leq q_j$.[2]
   - A reshuffle $\lambda$ has a right adjoint $\rho$ if and only if

     $$\text{for all } j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b}), \text{ we have } j \cdot \lambda < \odot_{\top}, \text{ and} \tag{2.8}$$

     $$\text{for all } j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b}) \text{ with } j \cdot \lambda = \otimes_i, \text{ we have } a_i = \bigcirc \text{ (so } j \cdot \lambda = \odot_i). \tag{2.9}$$

     In this case,
     - for any $P_i \in \mathsf{OriDeg}^{\top}_{\mathrm{eq}}(\vec{a})$, we find $P_i \bullet \rho$ as the least oriented degree $Q_j \in \mathsf{OriDeg}^{\top}_{\mathrm{eq}}(\vec{b})$ such that $P_i \leq Q_j \bullet \lambda$;
     - for any $p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, c)$, we find $p_i \circ \rho$ as the least polarized degree $q_j \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{b}, c)$ such that $p_i \leq q_j \circ \lambda$.

*Proof.* First of all, throughout the statement of the theorem, the formulation in terms of oriented degrees is equivalent to the formulation in terms of polarized degrees with 'codomain' $\nearrow$ because we have isomorphisms of poset-valued presheaves $\mathrm{Hom}_{\mathsf{Pol}}(\sqcup, \nearrow) \cong \mathsf{Ori}$ and $\mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\sqcup, \nearrow) \cong \mathsf{OriDeg}^{\top}_{\mathrm{eq}}$. In both cases the isomorphism is given by the action $\rightarrowtriangle \bullet \sqcup$. Furthermore, the corresponding presheaves of polarities and polarized degrees with codomain $\bigcirc$ are pointwise full subposets of those with codomain $\nearrow$, so asking that the same conditions apply there, adds nothing to the statement as a whole.

1. First, assume that we have reshuffles $\lambda \in \mathrm{Hom}(\vec{a}, \vec{b})$ and $\rho \in \mathrm{Hom}(\vec{b}, \vec{a})$ satisfying eqs. (2.4) and (2.5). We show that $\lambda \dashv \rho$, i.e. that $\mathrm{id}_{\vec{a}} \leq \rho \circ \lambda$ and $\lambda \circ \rho \leq \mathrm{id}_{\vec{b}}$.
   To see the former, we need to show that for all $i \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a})$, we have $\oplus_i \leq (i \cdot \rho) \circ \lambda$:

$$\begin{aligned} \oplus_i \leq (i \cdot \rho) \circ \lambda \quad &\Leftrightarrow \quad \oplus_i \leq \oplus_i \circ \rho \circ \lambda && \text{(corollary 2.1.2°15)} \\ &\Leftrightarrow \quad \oplus_i \circ \rho \leq \oplus_i \circ \rho && \text{(eq. (2.5))} \\ &\Leftrightarrow \quad \text{true.} \end{aligned}$$

[2] Recall from corollary 2.1.2°15 that $j \cdot \lambda = \oplus_j \circ \lambda$.

To see the latter, we need to show that for all $j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b})$, we have $(j \cdot \lambda) \circ \rho \leq \oplus_j$:

$$\begin{aligned}
(j \cdot \lambda) \circ \rho \leq \oplus_j \quad &\Leftrightarrow \quad \oplus_j \circ \lambda \circ \rho \leq \oplus_j && \text{(corollary 2.1.2°15)} \\
&\Leftrightarrow \quad \oplus_j \circ \lambda \leq \oplus_j \circ \lambda && \text{(eq. (2.5))} \\
&\Leftrightarrow \quad \text{true.}
\end{aligned}$$

Conversely, assume that $\lambda \dashv \rho$. Then we have

$$\begin{aligned}
P_i \leq Q_j \bullet \lambda \quad &\Rightarrow \quad P_i \bullet \rho \leq (Q_j \bullet \lambda) \bullet \rho = Q_j \bullet (\lambda \circ \rho) \leq Q_j, \\
P_i \bullet \rho \leq Q_j \quad &\Rightarrow \quad P_i \leq P_i \bullet (\rho \circ \lambda) = (P_i \bullet \rho) \bullet \lambda \leq Q_j \bullet \lambda,
\end{aligned}$$

so all adjoint pairs satisfy eq. (2.4) and, equivalently, eq. (2.5).

2. Next, we move to the criteria to determine existence of a left/right adjoint and the algorithms for constructing them. The algorithms are just rephrasings of eqs. (2.4) and (2.5), so assuming that a left/right adjoint exists, they are clearly correct.
**Existence of left adjoint.** Assume given a reshuffle $\rho \in \mathrm{Hom}(\vec{b}, \vec{a})$. We would like to define in a first step a function (parametrized by $c \in \mathbb{A}$)

$$\ell_c : \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{b}, c) \to \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, c) : q_j \mapsto \max\left\{ p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, c) \,\middle|\, p_i \circ \rho \leq q_j \right\}.$$

A left adjoint $\lambda \dashv \rho$ exists if and only if this function is definable and is the action of some polarized reshuffle on polarized degrees. However, $\ell_c$ is only defined if the set of which we take the maximum[3], is non-empty[4]. As such, existence of a left adjoint requires that

$$\forall q_j \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{b}, c). \exists p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, c). p_i \circ \rho \leq q_j.$$

Without loss of generality, we can instantiate both $p_i$ and $q_j$ with the minimal element $\odot_{\mathrm{eq}}$, so existence of a left adjoint requires that $\mathrm{eq} \cdot \rho = \odot_{\mathrm{eq}} \circ \rho = \odot_{\mathrm{eq}}$ (eq. (2.6)).
We now assume that this is the case, so that $\ell_c$ is defined. It is then, by construction, right adjoint to $\sqcup \circ \rho$. We define a polarized reshuffle $\lambda \in \mathrm{Hom}(\vec{a}, \vec{b})$ by $j \cdot \lambda = \ell_{b_j}(\oplus_j) \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, b_j)$. We have to show that $\ell$ is the action of $\lambda$ if and only if eq. (2.7) holds.

- To prove 'only if', assume that eq. (2.7) does not hold (which is decidable). Then there exists some $i \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a})$ such that $i \cdot \rho = \circledast_j \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{b}, a_i)$ with $b_j = \nearrow$.
This implies that $\circledast_i \circ \rho = \circledast_j \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{b}, \nearrow)$ and hence, by the adjunction $\sqcup \circ \rho \dashv \ell$, that $\circledast_i \leq \ell_{\nearrow}(\circledast_j)$.
On the other hand, we have that $\otimes_i \circ \rho = \circledast_j \not\leq \oplus_j \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{b}, \nearrow)$ and hence, by the adjunction $\sqcup \circ \rho \dashv \ell$, that $\otimes_i \not\leq \ell_{\nearrow}(\oplus_j) = j \cdot \lambda \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, \nearrow)$. This is only possible if $\mathsf{deg}(j \cdot \lambda) < i$. But this degree is preserved by the action of $\circledast$.
We tie our findings together:
$$\mathsf{deg}(\circledast_j \circ \lambda) = \mathsf{deg}(j \cdot \lambda) < i = \mathsf{deg}(\circledast_i) \leq \mathsf{deg}(\ell_{\nearrow}(\circledast_j))$$
so that $\ell$ cannot be the action of $\lambda$.
- To prove 'if', assume that eq. (2.7) holds. We prove that $\ell_c$ is the action of $\lambda$. It is sufficient to prove this for $c = \nearrow$, since there is just one composition table for polarities (fig. 2.1) and no additional equalities are introduced if the codomain $c$ becomes $\bigcirc$. It is clear from the definition of $\ell$ that $\ell(\odot_{\top}) = \odot_{\top}$, and for all other degrees we have $\ell(\oplus_j) = \oplus_j \circ \lambda$ by definition of $\lambda$ and corollary 2.1.2°15.
$$\ominus_j \circ \lambda = \ominus \circ (j \cdot \lambda) = \ominus \circ \ell(\oplus_j)$$
(Definition of $\lambda$)

[3] Which differs from the join in that it needs to be one of the operands.
[4] In which case there is indeed a maximum, since the set is closed under joins, since $\sqcup \bullet \rho$ respects joins (proposition 2.1.2°18).

$$
\begin{aligned}
&= \ominus \circ \max\left\{ p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, b_j) \,\middle|\, p_i \circ \rho \leq \oplus_j \right\} \\
&\qquad \text{(Definition of } \ell\text{)} \\
&= \max\left\{ \ominus \circ p_i \,\middle|\, p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, b_j) \text{ with } p_i \circ \rho \leq \oplus_j \right\} \\
&\qquad (\ominus \circ \sqcup : \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, b_j) \to \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, \nearrow) \text{ preserves } \leq.) \\
&= \max\left\{ p'_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, \nearrow) \,\middle|\, \ominus \circ p'_i \circ \rho \leq \oplus_j \right\} \\
&\qquad \text{(By taking } p'_i := \ominus \circ p_i\text{)} \\
&= \max\left\{ p'_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, \nearrow) \,\middle|\, p'_i \circ \rho \leq \ominus_j \right\} \\
&\qquad \text{(Involution } \leftarrow \cdot \sqcup \text{ preserves and reflects } \leq.\text{)} \\
&= \ell(\ominus_j), \\
\otimes_j \circ \lambda &= (\oplus_j \circ \lambda) \wedge (\ominus_j \circ \lambda) \\
&\qquad \text{(By proposition 2.1.2°18)} \\
&= \ell_{\nearrow}(\oplus_j) \wedge \ell_{\nearrow}(\ominus_j) \\
&\qquad \text{(By previous results)} \\
&= \ell_{\nearrow}(\oplus_j \wedge \ominus_j) = \ell_{\nearrow}(\otimes_j) \\
&\qquad \text{(Right adjoints preserve meets),} \\
\circledast_j \circ \lambda &= (\oplus_j \circ \lambda) \vee (\ominus_j \circ \lambda) \\
&\qquad \text{(By proposition 2.1.2°18)} \\
&= \ell_{\nearrow}(\oplus_j) \vee \ell_{\nearrow}(\ominus_j) \\
&\qquad \text{(By previous results)} \\
&= \max\left\{ p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, \nearrow) \,\middle|\, p_i \circ \rho \leq \oplus_j \right\} \vee \max\left\{ p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, \nearrow) \,\middle|\, p_i \circ \rho \leq \ominus_j \right\} \\
&\qquad \text{(Definition of } \ell\text{)} \\
&= \bigvee\left\{ p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, \nearrow) \,\middle|\, p_i \circ \rho \leq \oplus_j \right\} \vee \bigvee\left\{ p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, \nearrow) \,\middle|\, p_i \circ \rho \leq \ominus_j \right\} \\
&\qquad \text{(The maxima exist, so they are equal to the joins (suprema).)} \\
&= \bigvee\left\{ p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, \nearrow) \,\middle|\, p_i \circ \rho \leq \oplus_j \right\} \cup \left\{ p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, \nearrow) \,\middle|\, p_i \circ \rho \leq \ominus_j \right\} \\
&\qquad \text{(Associativity.)} \\
&= \bigvee\left\{ p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, \nearrow) \,\middle|\, p_i \circ \rho \leq \oplus_j \text{ or } p_i \circ \rho \leq \ominus_j \right\} \\
&= \bigvee\left\{ p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, \nearrow) \,\middle|\, p_i \circ \rho \leq \circledast_j \right\} \\
&\qquad \text{(Using eq. (2.7).)} \\
&= \ell_{\nearrow}(\circledast_j). \\
&\qquad \text{(Definition and existence of } \ell.\text{)}
\end{aligned}
$$

**Existence of right adjoint.** Assume given a reshuffle $\lambda \in \mathrm{Hom}(\vec{a}, \vec{b})$. We would like to define in a first step a function (parametrized by $c \in \mathbb{A}$)

$$r_c : \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, c) \to \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{b}, c) : p_i \mapsto \min\left\{ q_j \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{b}, c) \,\middle|\, p_i \leq q_j \circ \lambda \right\}.$$

A right adjoint $\lambda \dashv \rho$ exists if and only if this function is definable and is the action of some polarized reshuffle. However, $r_c$ is only defined if the set of which we take the minimum, is non-empty. As such, existence of a right adjoint requires that

$$\forall p_i \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{a}, c). \exists q_j \in \mathsf{PolDeg}^{\top}_{\mathrm{eq}}(\vec{b}, c). p_i \leq q_j \cdot \lambda.$$

Without loss of generality, we can instantiate both $p_i$ and $q_j$ with the maximal element $\odot_\top$, in which case the resulting requirement $\odot_\top \leq \odot_\top \cdot \lambda$ is true by definition of the action of a polarized reshuffle.
We conclude that $r$ is always defined. It is then, by construction, left adjoint to $\sqcup \circ \lambda$. We define a polarized reshuffle $\rho \in \mathrm{Hom}(\vec{b}, \vec{a})$ by $i \cdot \rho = r_{a_i}(\oplus_i) \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{b}, a_i)$. We have to show that $r$ is the action of $\rho$ if and only if eqs. (2.8) and (2.9) hold.

- Assume that eq. (2.8) does not hold (which is decidable). Then there exists some $j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b})$ such that $\oplus_j \circ \lambda = j \cdot \lambda = \odot_\top \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{a}, b_j)$. Then by the adjunction $r \dashv \sqcup \circ \lambda$, we get $r_{b_j}(\odot_\top) \leq \oplus_j \neq \odot_\top = \odot_\top \circ \rho \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{b}, b_j)$, implying that $r$ cannot be the action of $\rho$.
- Assume that eq. (2.9) does not hold (which is decidable). Then there exists some $j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b})$ such that $j \cdot \lambda = \otimes_i \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{a}, b_j)$ with $a_i = {\times\!\!\!\!\nearrow}$.
  This implies that $\otimes_j \circ \lambda = \otimes_i \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{a}, {\times\!\!\!\!\nearrow})$ and hence, by the adjunction $r \dashv \sqcup \circ \lambda$, that $r_{\times\!\!\!\!\nearrow}(\otimes_i) \leq \otimes_j \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{b}, {\times\!\!\!\!\nearrow})$.
  This also implies that $\circledast_j \circ \lambda = \otimes_i \not\geq \oplus_i \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{a}, {\times\!\!\!\!\nearrow})$ and hence, by the adjunction $r \dashv \sqcup \circ \lambda$, that $i \cdot \rho = r_{\times\!\!\!\!\nearrow}(\oplus_i) \not\leq \circledast_j \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{b}, {\times\!\!\!\!\nearrow})$. This is only possible if $\mathsf{deg}(i \cdot \rho) > j$. But this degree is preserved by the action of $\otimes$.
  We tie our findings together:
  $$\mathsf{deg}(\otimes_i \circ \rho) = \mathsf{deg}(i \cdot \rho) > j = \mathsf{deg}(\otimes_j) \geq \mathsf{deg}(r_{\times\!\!\!\!\nearrow}(\otimes_i))$$
  so that $r$ cannot be the action of $\rho$.
- To prove 'if', assume that eqs. (2.8) and (2.9) hold. We prove that $r_c$ is the action of $\rho$. Again, it is sufficient to prove this for $c = {\times\!\!\!\!\nearrow}$. From eq. (2.8) and the definition of $r$, it follows that $r(\odot_\top) = \odot_\top$. For all other degrees we have $r(\oplus_i) = \oplus_i \circ \rho$ by definition of $\rho$ and corollary 2.1.2°15. Furthermore, entirely dually (w.r.t. $\leq$) to our earlier calculations, we have

$$
\begin{aligned}
&\ominus_i \circ \rho = r_{\times\!\!\!\!\nearrow}(\ominus_i) \\
&\qquad\qquad \text{(Analogous to dual situation.)} \\
&\circledast_i \circ \rho = (\oplus_i \circ \rho) \vee (\ominus_i \circ \rho) \\
&\qquad\qquad \text{(By proposition 2.1.2°18)} \\
&\quad = r_{\times\!\!\!\!\nearrow}(\oplus_i) \vee r_{\times\!\!\!\!\nearrow}(\ominus_i) \\
&\qquad\qquad \text{(By previous results)} \\
&\quad = r_{\times\!\!\!\!\nearrow}(\oplus_i \vee \ominus_i) = r_{\times\!\!\!\!\nearrow}(\circledast_i) \\
&\qquad\qquad \text{(Left adjoints preserve joins),} \\
&\otimes_i \circ \rho = (\oplus_i \circ \rho) \wedge (\ominus_i \circ \rho) \\
&\qquad\qquad \text{(By proposition 2.1.2°18)} \\
&\quad = r_{\times\!\!\!\!\nearrow}(\oplus_i) \wedge r_{\times\!\!\!\!\nearrow}(\ominus_i) \\
&\qquad\qquad \text{(By previous results)} \\
&\quad = \min\left\{ q_j \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{b}, {\times\!\!\!\!\nearrow}) \,\middle|\, \oplus_i \leq q_j \circ \lambda \right\} \wedge \min\left\{ q_j \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{b}, {\times\!\!\!\!\nearrow}) \,\middle|\, \ominus_i \leq q_j \circ \lambda \right\} \\
&\quad = \bigwedge\left\{ q_j \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{b}, {\times\!\!\!\!\nearrow}) \,\middle|\, \oplus_i \leq q_j \circ \lambda \right\} \wedge \bigwedge\left\{ q_j \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{b}, {\times\!\!\!\!\nearrow}) \,\middle|\, \ominus_i \leq q_j \circ \lambda \right\} \\
&\qquad\qquad \text{(The minima exist, so they are equal to the meets (infima).)} \\
&\quad = \bigwedge\left\{ q_j \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{b}, {\times\!\!\!\!\nearrow}) \,\middle|\, \oplus_i \leq q_j \circ \lambda \right\} \cup \left\{ q_j \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{b}, {\times\!\!\!\!\nearrow}) \,\middle|\, \ominus_i \leq q_j \circ \lambda \right\} \\
&\qquad\qquad \text{(Associativity.)} \\
&\quad = \bigwedge\left\{ q_j \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{b}, {\times\!\!\!\!\nearrow}) \,\middle|\, \oplus_i \leq q_j \circ \lambda \text{ or } \ominus_i \leq q_j \right\} \\
&\quad = \bigwedge\left\{ q_j \in \mathsf{PolDeg}^\top_{\mathrm{eq}}(\vec{b}, {\times\!\!\!\!\nearrow}) \,\middle|\, \otimes_i \leq q_j \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad \text{(Using eq. (2.9).)} \\
&= r_{\nearrow}(\otimes_i). \\
&\qquad \text{(Definition and existence of } r.\text{)}
\end{aligned}
$$
□

**Definition 2.1.3°3.** Let $\vec{a}$ be an anpolarity mask and $i \in \mathsf{Deg}^{\top}_{\mathrm{eq}}(\vec{a})$.

1. If $i < \top$, write $\uparrow(i) := \min\left\{ i' \in \mathsf{Deg}^{\top}_{\mathrm{eq}}(\vec{a}) \,\middle|\, i' > i \right\}$; we call this the **successor** of $i$.
2. If $i > \mathrm{eq}$, write $\downarrow(i) := \max\left\{ i' \in \mathsf{Deg}^{\top}_{\mathrm{eq}}(\vec{a}) \,\middle|\, i' < i \right\}$; we call this the **predecessor** of $i$.

**Theorem 2.1.3°4.** Let $\tau \in \mathrm{Hom}(\vec{a}, \vec{b})$ be a polarized reshuffle with left adjoint $\rho \in \mathrm{Hom}(\vec{b}, \vec{a})$. Then $\rho$ has a further left adjoint $\lambda \dashv \rho \dashv \tau$ if and only if

$$\mathsf{deg}(\uparrow(\mathrm{eq}) \cdot \tau) > \mathrm{eq}, \text{ and} \tag{2.10}$$

$$\forall j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b}), i \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a}), \text{ if } j \cdot \tau \leq \otimes_i \text{ and } \mathsf{deg}(\uparrow(j) \cdot \tau) > i \text{ then } b_j = \bigcirc. \tag{2.11}$$

In words, eq. (2.11) says that:

- The last mention of a symmetric degree $i \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a})$ by $\tau$ is at a symmetric degree $j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b})$;
- The last mention of a directed degree $i \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a})$ (hence $i \neq \mathrm{eq}$) is either:
  - of the form $\otimes_i$ at a symmetric degree $j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b})$;
  - or of the form $\oplus_i$ or $\ominus_i$ at an arbitrary degree $j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b})$;
  - it already could not be of the form $\circledast_i$ as per eq. (2.7);
- If a degree $i \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a})$ is never mentioned, then the last mention of a degree less than $i$ is at a symmetric degree $j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b})$.

*Proof.* We show that eq. (2.10) ⇔ eq. (2.6) and eq. (2.11) ⇔ eq. (2.7).

Starting from eq. (2.6), we have

$$
\begin{aligned}
& \mathrm{eq} \cdot \rho = \odot_{\mathrm{eq}} \\
\Leftrightarrow\quad & \mathrm{eq} \cdot \rho < \otimes_{\uparrow(\mathrm{eq})} \\
\Leftrightarrow\quad & \mathrm{eq} \cdot \rho < \oplus_{\uparrow(\mathrm{eq})} && \text{(Case distinction on } b_{\uparrow(\mathrm{eq})}\text{; } \rho \text{ satisfies eq. (2.9))} \\
\Leftrightarrow\quad & \odot_{\mathrm{eq}} \circ \rho \not\geq \oplus_{\uparrow(\mathrm{eq})} && \text{(By corollary 2.1.2°15. Sym. pol. degrees are always comparable)} \\
\Leftrightarrow\quad & \odot_{\mathrm{eq}} \not\geq \oplus_{\uparrow(\mathrm{eq})} \circ \tau && (\sqcup \circ \tau \dashv \sqcup \circ \rho) \\
\Leftrightarrow\quad & \odot_{\mathrm{eq}} < \uparrow(\mathrm{eq}) \cdot \tau && \text{(By corollary 2.1.2°15. Sym. pol. degrees are always comparable)} \\
\Leftrightarrow\quad & \mathrm{eq} < \mathsf{deg}(\uparrow(\mathrm{eq}) \cdot \tau).
\end{aligned}
$$

This gives eq. (2.10).

Recall eq. (2.7):

$$\text{for all } i \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a}) \text{ with } i \cdot \rho = \circledast_j, \text{ we have } b_j = \bigcirc \text{ (so } i \cdot \rho = \odot_j).$$

We can rewrite the condition as $\circledast_j \leq i \cdot \rho$ **AND** $\circledast_j \geq i \cdot \rho$. We have

$$
\begin{aligned}
& \circledast_j \leq i \cdot \rho \\
\Leftrightarrow\quad & \circledast_j \circ \tau \leq \oplus_i && (\sqcup \circ \tau \dashv \sqcup \circ \rho\text{; corollary 2.1.2°15)} \\
\Leftrightarrow\quad & (\oplus_j \circ \tau) \vee (\ominus_j \circ \tau) \leq \oplus_i && \text{(Proposition 2.1.2°18)} \\
\Leftrightarrow\quad & \oplus_j \circ \tau \leq \oplus_i \text{ and } \ominus_j \circ \tau \leq \oplus_i \\
\Leftrightarrow\quad & \oplus_j \circ \tau \leq \oplus_i \text{ and } \oplus_j \circ \tau \leq \ominus_i && \text{(Apply } \ominus \circ \sqcup \text{ to both sides)} \\
\Leftrightarrow\quad & j \cdot \tau \leq \otimes_i
\end{aligned}
$$

and

$$
\begin{array}{lll}
& \circledast_j \geq i \cdot \rho & \\
\Leftrightarrow & \otimes_{\uparrow(j)} > i \cdot \rho & \\
\Leftrightarrow & \otimes_{\uparrow(j)} \not\leq \oplus_i \circ \rho & \text{(By corollary 2.1.2°15. Sym. pol. degrees are always comparable)} \\
\Leftrightarrow & \otimes_{\uparrow(j)} \circ \tau \not\leq \oplus_i & (\sqcup \circ \tau \dashv \sqcup \circ \rho) \\
\Leftrightarrow & \otimes \circ (\uparrow(j) \cdot \tau) > \oplus_i & \text{(By eq. (2.1). Sym. pol. degrees are always comparable)} \\
\Leftrightarrow & \otimes_{\mathsf{deg}(\uparrow(j)\cdot\tau)} > \oplus_i & \text{(Case distinction on } a_{\mathsf{deg}(\uparrow(j)\cdot\tau)}\text{; } \tau \text{ satisfies eq. (2.7))} \\
\Leftrightarrow & \mathsf{deg}(\uparrow(j) \cdot \tau) > i &
\end{array}
$$

Thus, for any $i$, if both of the above conditions for $\tau$ are satisfied, we conclude that $b_j = \bigcirc$. This gives eq. (2.11). □

**Theorem 2.1.3°5.** Let $\kappa \in \mathrm{Hom}(\vec{b}, \vec{a})$ be a polarized reshuffle with right adjoint $\lambda \in \mathrm{Hom}(\vec{a}, \vec{b})$. Then $\lambda$ has a further right adjoint $\kappa \dashv \lambda \dashv \rho$ if and only if

$$\downarrow(\top) \cdot \kappa > \otimes_{\downarrow(\top)} \tag{2.12}$$

$$\forall j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b}), i \in \mathsf{Deg}(\vec{a}), \text{ if } \mathsf{deg}(\downarrow(i) \cdot \kappa) < j \text{ and } i \cdot \kappa \geq \circledast_j \text{ then } a_i = \bigcirc. \tag{2.13}$$

In words, eq. (2.13) says that:

- The first mention of a symmetric degree $j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b})$ by $\kappa$ is at a symmetric degree $i \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a})$;
- The first mention of a directed degree $j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b})$ by $\kappa$ is either:
  - of the form $\circledast_j$ at a symmetric degree $i \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a})$;
  - or of the form $\oplus_j$ or $\ominus_j$ at an arbitrary degree $i \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a})$;
  - it already could not be of the form $\otimes_j$ as per eq. (2.9);
- If a degree $j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b})$ is never mentioned, then the first mention of degree greater than $j$ is at a symmetric degree $i \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a})$.

*Proof.* We show that eq. (2.12) ⇔ eq. (2.8) and eq. (2.13) ⇔ eq. (2.9).

Note that eq. (2.8) can equivalently be stated only for $j = \downarrow(\top)$. Then we have

$$
\begin{array}{lll}
& \downarrow(\top) \cdot \lambda < \odot_\top & \\
\Leftrightarrow & \oplus_{\downarrow(\top)} \circ \lambda \leq \circledast_{\downarrow(\top)} & \text{(Corollary 2.1.2°15)} \\
\Leftrightarrow & \oplus_{\downarrow(\top)} \leq \circledast_{\downarrow(\top)} \circ \kappa & (\sqcup \circ \lambda \dashv \sqcup \circ \kappa) \\
\Leftrightarrow & \otimes_{\downarrow(\top)} < \downarrow(\top) \cdot \kappa & \text{(Case distinction).}
\end{array}
$$

Recall eq. (2.9):

$$\text{for all } j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b}) \text{ with } j \cdot \lambda = \otimes_i, \text{ we have } a_i = \bigcirc \text{ (so } j \cdot \lambda = \odot_i\text{).}$$

We can rewrite the condition as $j \cdot \lambda \leq \otimes_i$ **AND** $\otimes_i \leq j \cdot \lambda$. We have

$$
\begin{array}{lll}
& j \cdot \lambda \leq \otimes_i & \\
\Leftrightarrow & \oplus_j \leq \otimes_i \circ \kappa & (\sqcup \circ \lambda \dashv \sqcup \circ \kappa\text{; corollary 2.1.2°15)} \\
\Leftrightarrow & \oplus_j \leq (\oplus_i \circ \kappa) \wedge (\ominus_i \circ \kappa) & \text{(proposition 2.1.2°18)} \\
\Leftrightarrow & \oplus_j \leq \oplus_i \circ \kappa \text{ and } \oplus_j \leq \ominus_i \circ \kappa & \\
\Leftrightarrow & \oplus_j \leq \oplus_i \circ \kappa \text{ and } \ominus_j \leq \oplus_i \circ \kappa & \text{(Apply } \ominus \circ \sqcup \text{ to both sides)} \\
\Leftrightarrow & \circledast_j \leq i \cdot \kappa &
\end{array}
$$

and

$$j \cdot \lambda \geq \otimes_i$$

$$
\begin{aligned}
&\Leftrightarrow \quad j \cdot \lambda > \circledast_{\downarrow(i)} && \text{(WLOG, } i > \mathrm{eq}\text{, since } a_{\mathrm{eq}} = \bigcirc \text{ regardless)}\\
&\Leftrightarrow \quad \oplus_j \circ \lambda \not\leq \circledast_{\downarrow(i)} && \text{(By corollary 2.1.2°15. Sym. pol. degrees are always comparable)}\\
&\Leftrightarrow \quad \oplus_j \not\leq \circledast_{\downarrow(i)} \circ \kappa && (\sqcup \circ \lambda \dashv \sqcup \circ \kappa)\\
&\Leftrightarrow \quad \oplus_j > \circledast \circ (\downarrow(i) \cdot \kappa) && \text{(By eq. (2.1). Sym. pol. degrees are always comparable)}\\
&\Leftrightarrow \quad \oplus_j > \circledast_{\mathsf{deg}(\downarrow(i)\cdot\kappa)} && \text{(Case distinction on } b_{\mathsf{deg}(\downarrow(i)\cdot\kappa)}\text{; } \kappa \text{ satisfies eq. (2.9))}\\
&\Leftrightarrow \quad j > \mathsf{deg}(\downarrow(i) \cdot \kappa).
\end{aligned}
$$

Thus, for any $j \in \mathsf{Deg}(\vec{b})$, if both of the above conditions for $\kappa$ are satisfied, we conclude that $a_i = \bigcirc$. This gives eq. (2.13). □

### 2.1.3 (b) Examples

**Example 2.1.3°6.** Let $a_i = \nearrow\!\!\!\!\times$. Then we have a self-inverse and therefore self-adjoint polarized reshuffle $\mathsf{Op}_i : \vec{a} \to \vec{a}$ defined by $i \cdot \mathsf{Op}_i = \ominus_i$ and $j \cdot \mathsf{Op}_i = \oplus_j$ for all $i \neq j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a})$.

**Example 2.1.3°7.** Let $\vec{a} \sqsubset_i \vec{b}$ (definition 1.2.1°8). Then we have a triple of adjoint polarized reshuffles

$$\mathsf{F}_i^{\bigcirc} \dashv \mathsf{U}_i^{\bigcirc} \dashv \mathsf{Cof}_i^{\bigcirc}$$

where $\mathsf{U}_i^{\bigcirc} : \vec{a} \to \vec{b}$, defined by

$$
\begin{array}{llll}
i \cdot \mathsf{F}_i^{\bigcirc} = \circledast_i & i \cdot \mathsf{U}_i^{\bigcirc} = \odot_i & i \cdot \mathsf{Cof}_i^{\bigcirc} = \otimes_i & \\
j \cdot \mathsf{F}_i^{\bigcirc} = \oplus_i & j \cdot \mathsf{U}_i^{\bigcirc} = \oplus_i & j \cdot \mathsf{Cof}_i^{\bigcirc} = \oplus_i & \text{for } i \neq j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a}) = \mathsf{Deg}_{\mathrm{eq}}(\vec{b})
\end{array}
$$

From theorem 2.1.3°2, we conclude that no further left or right adjoints exist. We have

$$\mathsf{F}_i^{\bigcirc} \circ \mathsf{U}_i^{\bigcirc} = \mathrm{id}, \qquad \mathsf{Cof}_i^{\bigcirc} \circ \mathsf{U}_i^{\bigcirc} = \mathrm{id}_{\vec{a}}.$$

Thus, we get an idempotent monad $\mathsf{Clo}_i^{\bigcirc} = \mathsf{U}_i^{\bigcirc} \circ \mathsf{F}_i^{\bigcirc}$ and an idempotent comonad $\mathsf{Int}_i^{\bigcirc} = \mathsf{U}_i^{\bigcirc} \circ \mathsf{F}_i^{\bigcirc}$ such that $\mathsf{Clo}_i^{\bigcirc} \dashv \mathsf{Int}_i^{\bigcirc}$. Both are given by

$$
\begin{array}{lll}
i \cdot \mathsf{Clo}_i^{\bigcirc} = \circledast_i & i \cdot \mathsf{Int}_i^{\bigcirc} = \otimes_i & \\
j \cdot \mathsf{Clo}_i^{\bigcirc} = \oplus_i & j \cdot \mathsf{Int}_i^{\bigcirc} = \oplus_i & \text{for } i \neq j \in \mathsf{Deg}_{\mathrm{eq}}(\vec{b})
\end{array}
$$

**Example 2.1.3°8.** Let $\vec{b}$ be obtained from $\vec{a}$ by deleting the anpolarity with degree $i$. Let $n := \mathsf{len}(\vec{a}) = \mathsf{len}(\vec{b}) + 1$. We define the following polarized reshuffle, which forgets the $i$-jet relation:

$$\sqcup_i = \langle \odot_{\mathrm{eq}} \mid \oplus_0, \ldots, \oplus_{i-1}, \oplus_{i+1}, \ldots, \oplus_{n-1} \rangle : \vec{a} \to \vec{b}$$

- If $i = 0$, then $\sqcup_0$ has two left adjoints $\sqcap_0 \dashv \triangle_0 \dashv \sqcup_0$ with

$$
\begin{aligned}
\sqcap_0 &= \langle \circledast_0 \mid \oplus_1, \ldots, \oplus_{n-1} \rangle && : \vec{a} \to \vec{b},\\
\dashv \triangle_0 &= \langle \odot_{\mathrm{eq}} \mid \odot_{\mathrm{eq}}, \oplus_0, \ldots, \oplus_{n-2} \rangle && : \vec{b} \to \vec{a},\\
\dashv \sqcup_0 &= \langle \odot_{\mathrm{eq}} \mid \oplus_1, \ldots, \oplus_{n-1} \rangle && : \vec{a} \to \vec{b}.
\end{aligned}
$$

  By theorem 2.1.3°2, $\sqcap_0$ has no further left adjoints. We have $\sqcap_0 \triangle_0 = \sqcup_0 \triangle_0 = \mathrm{id}_{\vec{b}}$. We get adjoint idempotent (co)monads

$$
\begin{aligned}
\int_0 = \triangle_0 \sqcap_0 &= \langle \circledast_0 \mid \circledast_0, \oplus_1, \ldots, \oplus_{n-1} \rangle && : \vec{a} \to \vec{a},\\
\dashv \flat_0 = \triangle_0 \sqcup_0 &= \langle \odot_{\mathrm{eq}} \mid \odot_{\mathrm{eq}}, \oplus_1, \ldots, \oplus_{n-1} \rangle : \vec{a} \to \vec{a}.
\end{aligned}
$$

- If $i > 0$, then $\sqcup_i$ has a left adjoint $\triangle_i \dashv \sqcup_i$ where

$$\triangle_i = \langle \odot_{\mathrm{eq}} \mid \oplus_0, \ldots, \oplus_{i-1}, \circledast_{i-1}, \oplus_i, \ldots, \oplus_{n-2} \rangle : \vec{b} \to \vec{a}.$$

  We have $\sqcup_i \triangle_i = \mathrm{id}_{\vec{b}}$. We get an idempotent comonad

$$\flat_i = \triangle_i \sqcup_i = \langle \odot_{\mathrm{eq}} \mid \oplus_0, \ldots, \oplus_{i-1}, \circledast_{i-1}, \oplus_{i+1}, \ldots, \oplus_{n-1} \rangle : \vec{a} \to \vec{a}$$

  - If $a_{i-1} = b_{i-1} = \not\nearrow$, then $\triangle_i$ has no further left adjoints.
  - If $a_{i-1} = b_{i-1} = \bigcirc$, then we have $\sqcap_i \dashv \triangle_i \dashv \sqcup_i$ with

$$\begin{aligned}
\sqcap_i &= \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{i-2}, \circledast_i, \oplus_{i+1}, \dots, \oplus_{n-1} \rangle &&: \vec{a} \to \vec{b}, \\
\dashv \triangle_i &= \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{i-2}, \odot_{i-1}, \odot_{i-1}, \oplus_i, \dots, \oplus_{n-2} \rangle &&: \vec{b} \to \vec{a}, \\
\dashv \sqcup_i &= \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{i-2}, \odot_{i-1}, \oplus_{i+1}, \dots, \oplus_{n-1} \rangle &&: \vec{a} \to \vec{b}.
\end{aligned}$$

    We have $\sqcap_i \triangle_i = \sqcup_i \triangle_i = \mathrm{id}_{\vec{b}}$. We get adjoint idempotent (co)monads

$$\begin{aligned}
\smallint_i = \triangle_i \sqcap_i &= \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{i-2}, \circledast_i, \circledast_i, \oplus_{i+1}, \dots, \oplus_{n-1} \rangle &&: \vec{a} \to \vec{a}, \\
\dashv \flat_i = \triangle_i \sqcup_i &= \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{i-2}, \odot_{i-1}, \odot_{i-1}, \oplus_{i+1}, \dots, \oplus_{n-1} \rangle &&: \vec{a} \to \vec{a}
\end{aligned}$$

    * If $a_i = \not\nearrow$, then there are no further left adjoints.
    * If $a_i = \bigcirc$, then $\sqcap_i = \sqcup_{i-1}$ and we can continue.

- If $i = n-1$, then $\sqcup_{n-1}$ has a right adjoint $\sqcup_{n-1} \dashv \triangledown_{n-1}$ with

$$\begin{aligned}
\sqcup_{n-1} &= \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{n-2} \rangle &&: \vec{a} \to \vec{b}, \\
\dashv \triangledown_{n-1} &= \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{n-2}, \odot_\top \rangle &&: \vec{b} \to \vec{a},
\end{aligned}$$

  and there are no further right adjoints. We have $\sqcup_{n-1} \triangledown_{n-1} = \mathrm{id}_{\vec{b}}$. We get an idempotent monad

$$\sharp_{n-1} = \triangledown_{n-1} \sqcup_{n-1} = \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{n-2}, \odot_\top \rangle : \vec{a} \to \vec{a}.$$

- If $i < n-1$, then $\sqcup_i$ has a right adjoint $\sqcup_i \dashv \triangledown_i$ with

$$\triangledown_i = \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{i-1}, \otimes_i, \oplus_i, \dots, \oplus_{n-2} \rangle : \vec{b} \to \vec{a}.$$

  We have $\sqcup_i \triangledown_i = \mathrm{id}_{\vec{b}}$. We get an idempotent monad

$$\sharp_i = \triangledown_i \sqcup_i = \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{i-1}, \otimes_{i+1}, \oplus_{i+1}, \dots, \oplus_{n-1} \rangle : \vec{a} \to \vec{a}.$$

  - If $a_{i+1} = b_i = \not\nearrow$, then there are no further right adjoints.
  - If $a_{i+1} = b_i = \bigcirc$, then we have $\sqcup_i \dashv \triangledown_i \dashv \boxminus_i$ with

$$\begin{aligned}
\sqcup_i &= \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{i-1}, \odot_{i+1}, \oplus_{i+2}, \dots, \oplus_{n-1} \rangle &&: \vec{a} \to \vec{b}, \\
\dashv \triangledown_i &= \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{i-1}, \odot_i, \odot_i, \oplus_{i+1}, \dots, \oplus_{n-2} \rangle &&: \vec{b} \to \vec{a}, \\
\dashv \boxminus_i &= \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{i-1}, \otimes_i, \oplus_{i+2}, \dots, \oplus_{n-1} \rangle &&: \vec{a} \to \vec{b}.
\end{aligned}$$

    We have $\sqcup_i \triangledown_i = \boxminus_i \triangledown_i = \mathrm{id}_{\vec{b}}$. We get adjoint idempotent (co)monads

$$\begin{aligned}
\sharp_i = \triangledown_i \sqcup_i &= \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{i-1}, \odot_{i+1}, \odot_{i+1}, \oplus_{i+2}, \dots, \oplus_{n-1} \rangle &&: \vec{a} \to \vec{a}, \\
\dashv \P_i = \triangledown_i \boxminus_i &= \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \dots, \oplus_{i-1}, \otimes_i, \otimes_i, \oplus_{i+2}, \dots, \oplus_{n-1} \rangle &&: \vec{a} \to \vec{a}.
\end{aligned}$$

    * If $a_i = \not\nearrow$, then there are no further right adjoints.
    * If $a_i = \bigcirc$, then

$$\begin{array}{ccccccccc}
\triangle_i & \dashv & \sqcup_i & \dashv & \triangledown_i & \dashv & \boxminus_i & & \\
 & & \parallel & & \parallel & & \parallel & & \\
 & & \sqcap_{i+1} & \dashv & \triangle_{i+1} & \dashv & \sqcup_{i+1} & \dashv & \triangledown_{i+1},
\end{array}$$

$$\begin{array}{ccccccc}
\flat_i & \dashv & \sharp_i & \dashv & \P_i & & \\
 & & \parallel & & \parallel & & \\
 & & \smallint_{i+1} & \dashv & \flat_{i+1} & \dashv & \sharp_{i+1},
\end{array}$$

      and we can continue.

**Example 2.1.3°9.** [5] We use the notations from section 3.1. Let $\vec{a} = [\nearrow\!\!\!\swarrow, \nearrow\!\!\!\swarrow]^{\mathsf{PJB};\text{-}} = [\bigcirc, \nearrow\!\!\!\swarrow, \bigcirc, \bigcirc, \nearrow\!\!\!\swarrow, \bigcirc]$ and $\vec{b} = [\nearrow\!\!\!\swarrow]^{\mathsf{PJB};\text{-}} = [\bigcirc, \nearrow\!\!\!\swarrow, \bigcirc]$. We have the following chain of adjoint reshuffles, now annotated with big degrees, which cannot be extended on either side:

$$\begin{aligned}
\sqcap_0^{\mathsf{PJB}} &= \langle \odot_0^{\mathsf{B}} \mid \odot_1^{\mathsf{P}}, \oplus_1^{\mathsf{J}}, \odot_1^{\mathsf{B}} \rangle && : \vec{a} \to \vec{b}, \\
\dashv \Delta_0^{\mathsf{PJB}} &= \langle \odot_{\mathrm{eq}} \mid \odot_{\mathrm{eq}}, \odot_{\mathrm{eq}}, \odot_{\mathrm{eq}}, \odot_0^{\mathsf{P}}, \oplus_0^{\mathsf{J}}, \odot_0^{\mathsf{B}} \rangle && : \vec{b} \to \vec{a}, \\
\dashv \sqcap_{0\;\;1}^{\mathsf{JB};\mathsf{P}} = \sqcup_0^{\mathsf{PJB}} &= \langle \odot_{\mathrm{eq}} \mid \odot_1^{\mathsf{P}}, \oplus_1^{\mathsf{J}}, \odot_1^{\mathsf{B}} \rangle && : \vec{a} \to \vec{b}, \\
\dashv \Delta_{0\;\;1}^{\mathsf{JB};\mathsf{P}} = \nabla_0^{\mathsf{PJB}} &= \langle \odot_{\mathrm{eq}} \mid \odot_0^{\mathsf{P}}, \odot_0^{\mathsf{P}}, \odot_0^{\mathsf{P}}, \odot_0^{\mathsf{P}}, \oplus_0^{\mathsf{J}}, \odot_0^{\mathsf{B}} \rangle && : \vec{b} \to \vec{a}, \\
\dashv \sqcup_{0\;\;1}^{\mathsf{JB};\mathsf{P}} = \boxminus_0^{\mathsf{PJB}} &= \langle \odot_{\mathrm{eq}} \mid \odot_0^{\mathsf{P}}, \oplus_1^{\mathsf{J}}, \odot_1^{\mathsf{B}} \rangle && : \vec{a} \to \vec{b}, \\
\dashv \nabla_{0\;\;1}^{\mathsf{JB};\mathsf{P}} = &\quad\;\; \langle \odot_{\mathrm{eq}} \mid \odot_0^{\mathsf{P}}, \otimes_0^{\mathsf{J}}, \otimes_0^{\mathsf{J}}, \otimes_0^{\mathsf{J}}, \oplus_0^{\mathsf{J}}, \odot_0^{\mathsf{B}} \rangle && : \vec{b} \to \vec{a}.
\end{aligned}$$

The function type with modality $\sqcup_0^{\mathsf{PJB}}$ will take a limit of a covariant functor, and could be called ‘covariant naturality’. The modality $\nabla_0^{\mathsf{PJB}}$ will be used to lift structure inhabiting a covariant functor to $\mathsf{Type}^{\vec{b}}$, to mode $\vec{a}$, e.g. to build the category of pointed types. It could be called ‘covariant structurality’, generalizing the structural modality in Degrees of Relatedness [ND18].[6]

#### 2.1.3 (c) Additional Results

**Proposition 2.1.3°10.** Let $\mu : \vec{a} \to \vec{b}$ be a polarized reshuffle, $a_i = b_j = \nearrow\!\!\!\swarrow$, and let $j \cdot \mu \in \{\oplus_i, \ominus_i\}$. Then $\mathsf{Op}_j \circ \mu = \mu \circ \mathsf{Op}_i$.

*Proof.* We prove this if $j \cdot \mu = \oplus_i$; otherwise we can set $\mu' = \mathsf{Op}_j \circ \mu$. Clearly, we have:

$$j \cdot (\mathsf{Op}_j \circ \mu) = \ominus_j \circ \mu = \ominus_i = (j \cdot \mu) \circ \mathsf{Op}_i.$$

Now if $j' \neq j$, then we have

$$\begin{aligned}
j' \cdot (\mathsf{Op}_j \circ \mu) &= \oplus_{j'} \circ \mu = j' \cdot \mu, \\
j' \cdot (\mu \circ \mathsf{Op}_j) &= (j' \cdot \mu) \circ \mathsf{Op}_j,
\end{aligned}$$

so we have to prove that $(j' \cdot \mu) \circ \mathsf{Op}_j = j' \cdot \mu$. This is the case if either $\mathsf{deg}(j' \cdot \mu) \neq i$ or $\mathsf{pol}(j' \cdot \mu)$ is symmetric. But that follows from the fact that $j' \cdot \mu \ll j \cdot \mu$ or $j \cdot \mu \ll j' \cdot \mu$. ☐

**Proposition 2.1.3°11.** Let $\lambda : \vec{a} \to \vec{b}$ be a polarized reshuffle with adjoint $\lambda \dashv \rho$. Let $a_i = b_j = \nearrow\!\!\!\swarrow$. Then we have:

- $j \cdot \lambda = \oplus_i \quad \Leftrightarrow \quad i \cdot \rho = \oplus_j$,
- $j \cdot \lambda = \ominus_i \quad \Leftrightarrow \quad i \cdot \rho = \ominus_j$.

*Proof.* We will prove the first equivalence. The second one then follows by considering $\lambda' = \mathsf{Op}_j \circ \lambda = \lambda \circ \mathsf{Op}_i$ and its adjoint $\rho' = \mathsf{Op}_i \circ \rho = \rho \circ \mathsf{Op}_j$.

We decompose the equality as $\oplus_i \leq j \cdot \lambda < \circledast_i$. For the first inequality, we have

$$\oplus_i \leq j \cdot \lambda \quad \Leftrightarrow \quad \oplus_i \leq \oplus_j \circ \lambda \quad \Leftrightarrow \quad \oplus_i \circ \rho \leq \oplus_j \quad \Leftrightarrow \quad i \cdot \rho \leq \oplus_j.$$

On the other hand, we have

$$\begin{aligned}
& \circledast_i > j \cdot \lambda \\
\Leftrightarrow\quad & \circledast_i \not\leq \oplus_j \circ \lambda && \text{(Corollary 2.1.2°15. Sym. pol. degrees are always comparable)} \\
\Leftrightarrow\quad & \circledast_i \circ \rho \not\leq \oplus_j && (\sqcup \circ \rho \dashv \sqcup \circ \lambda) \\
\Leftrightarrow\quad & \circledast_i \circ \rho > \oplus_j && \text{Sym. pol. degrees are always comparable).}
\end{aligned}$$

Since $\mathsf{deg}(\circledast_i \circ \rho) = \mathsf{deg}(i \cdot \rho)$, we can conclude that both must be equal to $j$. Then from $i \cdot \rho \leq \oplus_j$, it follows that $i \cdot \rho \in \{\otimes_j, \oplus_j\}$. Then $\circledast_i \circ \rho > \oplus_j$ rules out $\otimes_j$, so we conclude that $i \cdot \rho = \oplus_j$. ☐

[5] Make sure to refer correctly.
[6] Not 100% sure right now.

## 2.2 Semantics on Jet Sets and Jet Setoids

**Notation 2.2.0°1.** Let $\mathcal{V}$ be a monoidal category. We write $\mathcal{V}\mathsf{Cat}$ for the category of $\mathcal{V}$-enriched categories.

Let $F : \mathcal{V} \to \mathcal{W}$ be a monoidal functor. Then we write $F_1 : \mathcal{V}\mathsf{Cat} \to \mathcal{W}\mathsf{Cat}$ for the functor that applies $F$ Hom-wise.

**Definition 2.2.0°2.** A **setoid** is a set equipped with an equivalence relation $\frown_{\mathrm{eq}}$. A **setoid morphism** is a function between setoids respecting $\frown_{\mathrm{eq}}$. This defines the category $\mathsf{Setoid}$ of setoids.

A setoid is called **discrete** if the equivalence relation is the equality relation.

$\mathsf{Setoid}$ can be regarded as a $\mathsf{Setoid}$-enriched category by considering setoid morphisms equivalent if they are pointwise equivalent.

**Definition 2.2.0°3.** We have an adjoint triple $\mathbf{Q} \dashv \mathbf{D} \dashv \mathbf{U}$ between the (ordinary) category of setoids and $\mathsf{Set}$:

- $\mathbf{Q} : \mathsf{Setoid} \to \mathsf{Set}$ quotients out the equivalence relation,
- $\mathbf{D} : \mathsf{Set} \to \mathsf{Setoid}$ equips a set with a discrete equivalence relation (i.e. the equality relation),
- $\mathbf{U} : \mathsf{Setoid} \to \mathsf{Set}$ forgets the equivalence relation.

We can now use $\mathbf{U}_1 : \mathsf{SetoidCat} \to \mathsf{Cat}$ to disambiguate between the setoid-enriched category of setoids $\mathsf{Setoid}$ and the ordinary category of setoids $\mathbf{U}_1\mathsf{Setoid}$.

**Corollary 2.2.0°4.** The *setoid-enriched* quotient and discrete functor $\mathbf{Q} : \mathsf{Setoid} \to \mathbf{D}_1\mathsf{Set}$ and $\mathbf{D} : \mathbf{D}_1\mathsf{Set} \to \mathsf{Setoid}$ form an equivalence of $\mathsf{Setoid}$-enriched categories (where $\mathbf{DQ}$ is only naturally *equivalent*, not naturally isomorphic, to the identity functor). □

**Definition 2.2.0°5.** We define the category $\mathsf{JetSetoid}(\vec{a})$ of $\vec{a}$**-jet-setoids** as the following strict pullback:

$$\begin{tikzcd}
\mathsf{JetSetoid}(\vec{a}) \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{JetSet}(\vec{a}) \arrow[d, "U"] \\
\mathsf{Setoid} \arrow[r, "\mathbf{Q}"'] & \mathsf{Set}
\end{tikzcd}$$

where $U$ sends a jet set to its carrier and $\mathbf{Q}$ quotients out the equivalence relation. In other words, a jet setoid $X$ is a setoid $X$ with a jet set structure on $\mathbf{Q}X$. However, rather than working with equivalence classes explicitly, we will pull back the jet set structure along $[\sqcup] : X \to \mathbf{Q}X$ and consider an equivalence-respecting jet set structure directly on $X$. A jet setoid morphism is then a setoid morphism $f : X \to Y$ such that $x_1 \rightarrowtriangle_i x_2$ for $x_1, x_2 \in X$ implies $f(x_1) \rightarrowtriangle_i f(x_2)$.

We turn $\mathsf{JetSetoid}(\vec{a})$ into a $\mathsf{Setoid}$-enriched category by declaring morphisms equivalent if the underlying setoid morphisms are equivalent.

**Corollary 2.2.0°6.** The adjunction $\mathbf{Q} \dashv \mathbf{D}$ of categories from definition 2.2.0°3 carries over to an adjunction $\mathbf{Q} \dashv \mathbf{D}$ between $\mathbf{U}_1\mathsf{JetSetoid}(\vec{a})$ and $\mathsf{JetSet}(\vec{a})$. □

**Corollary 2.2.0°7.** The equivalence of setoid-enriched categories from corollary 2.2.0°4 carries over to an equivalence of setoid-enriched categories $\mathsf{JetSetoid}(\vec{a}) \simeq \mathbf{D}_1\mathsf{JetSet}(\vec{a})$. □

**Definition 2.2.0°8.**
1. We define a strict[7] 2-functor $\mathsf{JetSetoid} : \mathsf{PoResh} \to \mathsf{SetoidCat}$ from the 2-category of polarized reshuffles to the 2-category of setoid-enriched categories:
   - Sending an object $\vec{a}$ to $\mathsf{JetSetoid}(\vec{a})$.
   - Sending a morphism $\mu : \vec{a} \to \vec{b}$ to $\mathsf{JetSetoid}(\mu) : \mathsf{JetSetoid}(\vec{a}) \to \mathsf{JetSetoid}(\vec{b})$ where $\mathsf{JetSetoid}(\mu)$ (more briefly denoted as $\mu$) sends a jet setoid $X$ to the jet setoid $\mu X$ whose
     - carrier is the carrier of $X$,
     - equivalence relation $\frown_{\mathrm{eq}}^{\mu X}$ is given by transitive closure of $\frown_{\mathrm{eq}} \bullet \mu$ on $X$,

[7] Assuming propositional extensionality.

  - $i$-jet relation $\multimapdotbothB_i^{\mu X}$ is given by the relation $\multimapdotbothB_i \cdot \mu$ on $X$, made stable under $\frown_{\mathrm{eq}}^{\mu X}$, i.e. we have $\multimapdotbothB_i^{\mu X} = (\frown_{\mathrm{eq}} \bullet \mu)^* \otimes (\multimapdotbothB_i \cdot \mu) \otimes (\frown_{\mathrm{eq}} \bullet \mu)^*$, where $\otimes$ is composition of relations via existential quantification.

  $\mathsf{JetSetoid}(\mu)$ is easily seen to have a functorial action on jet set morphisms which preserves identity and composition.

- Sending a unique proof of inequality $\mu \leq \nu : \vec{a} \to \vec{b}$ to an obvious natural transformation $[\mu \leq \nu\rangle : \mu \to \nu : \mathsf{JetSetoid}(\vec{a}) \to \mathsf{JetSetoid}(\vec{b})$ whose underlying function is the identity function.

2. By corollary 2.2.0°7, we obtain a pseudofunctor $\mathsf{JetSet} : \mathsf{PoResh} \to \mathsf{Cat}$:
   - Sending an object $\vec{a}$ to $\mathsf{JetSet}(\vec{a})$,
   - Sending a morphism $\mu : \vec{a} \to \vec{b}$ to $\mathsf{JetSet}(\mu) = \mathbf{Q} \circ \mathsf{JetSetoid}(\mu) \circ \mathbf{D} : \mathsf{JetSet}(\vec{a}) \to \mathsf{JetSet}(\vec{b})$,
   - …
3. Additionally, if we consider only modalities $\mu$ satisfying eq. (2.6) (i.e. those for which $\mathrm{eq} \cdot \mu = \odot_{\mathrm{eq}}$) and denote the restricted 2-poset as $\mathsf{PoResh}^{[\mathrm{eq}]}$, then we directly get a strict 2-functor $\mathsf{JetSet}^{[\mathrm{eq}]} : \mathsf{PoResh}^{[\mathrm{eq}]} \to \mathsf{Cat}$, equivalent to the pseudofunctor above:
   - Sending an object $\vec{a}$ to $\mathsf{JetSet}(\vec{a})$,
   - Sending a morphism $\mu : \vec{a} \to \vec{b}$ to $\mathsf{JetSet}^{[\mathrm{eq}]}(\mu) : \mathsf{JetSet}(\vec{a}) \to \mathsf{JetSet}(\vec{b})$ where $\mathsf{JetSet}^{[\mathrm{eq}]}(\mu)$ (more briefly denoted as $\mu$) sends a jet set $X$ to the jet set $\mu X$ whose
     - carrier is the carrier of $X$,
     - $i$-jet relation $\multimapdotbothB_i^{\mu X}$ is the relation $\multimapdotbothB_i \bullet \mu$ on $X$.
   - …

**Corollary 2.2.0°9.** Let $\mu : \vec{a} \to \vec{b}$ and $X \in \mathsf{JetSetoid}(\vec{a})$. For an arbitrary $Q_j \in \mathsf{OriDeg}_{\mathrm{eq}}^{\top}(\vec{b})$, the relation $Q_j$ on $\mu X$ is given by $(\frown_{\mathrm{eq}} \bullet \mu)^* \otimes (Q_j \bullet \mu) \otimes (\frown_{\mathrm{eq}} \bullet \mu)^*$. Here, $\frown_{\top}$ is interpreted as 'true'.

If $\mu$ has a left adjoint and $X \in \mathsf{JetSet}(\vec{a})$, then for an arbitrary $Q_j \in \mathsf{Ori}(\vec{b})$, the relation $Q_j$ on $\mu X$ is given by $Q_j \bullet \mu$. □

**Example 2.2.0°10.** The functor $\mathsf{Op}_i$ from definition 1.2.1°7 arises from the reshuffle in example 2.1.3°6 as $\mathsf{JetSet}^{[\mathrm{eq}]}(\mathsf{Op}_i)$.

**Example 2.2.0°11.** The functors $\mathsf{FSym}_i \dashv \mathsf{USym}_i \dashv \mathsf{CofSym}_i$ and $\mathsf{SymCl}_i \dashv \mathsf{SymInt}_i$ from definition 1.2.1°8 and proposition 1.2.1°9 arise from the reshuffles in example 2.1.3°7 as

$$\mathsf{JetSet}^{[\mathrm{eq}]}(\mathsf{F}_i^{\bigcirc}) \dashv \mathsf{JetSet}^{[\mathrm{eq}]}(\mathsf{U}_i^{\bigcirc}) \dashv \mathsf{JetSet}^{[\mathrm{eq}]}(\mathsf{Cof}_i^{\bigcirc}),$$

$$\mathsf{JetSet}^{[\mathrm{eq}]}(\mathsf{Clo}_i^{\bigcirc}) \dashv \mathsf{JetSet}^{[\mathrm{eq}]}(\mathsf{Int}_i^{\bigcirc}).$$

### 2.2.1 Intervals and Prisms

**Definition 2.2.1°1.** We view the intervals from definition 1.2.2°1 not just as jet sets but also as (discrete) jet setoids. We additionally introduce $(\!|\frown_{\mathrm{eq}}|\!)$, the jet setoid on $\{0,1\}$ generated by $0 \frown_{\mathrm{eq}} 1$, and $(\!|\frown_{\top}|\!)$ as the jet set or jet setoid on $\{0,1\}$ where the equivalence and all $i$-jet relations are reflexive.

**Proposition 2.2.1°2.** Let $\mu : \vec{a} \to \vec{b}$ be a polarized reshuffle with right adjoint $\mu \dashv \rho$. Then we have[8]

$$\mathsf{JetSetoid}(\mu)(\!|\multimapdotbothB_i|\!) = (\!|\multimapdotbothB_i \bullet \rho|\!), \quad \mathsf{JetSetoid}(\mu)(\!|\multimapdotbothA_i|\!) = (\!|\multimapdotbothA_i \bullet \rho|\!), \quad \mathsf{JetSetoid}(\mu)(\!|\multimapboth_i|\!) = (\!|\multimapboth_i \bullet \rho|\!). \tag{2.14}$$

*Proof.* By theorem 2.1.3°2, $P_i \bullet \rho$ is the least symbol (i.e. the strictest relation) $Q_j$ such that $P_i \leq Q_j \bullet \mu$, i.e. so that $0\ Q_j\ 1$ in $\mu(\!|P_i|\!)$. This means that the jet setoid structure of $\mu(\!|P_i|\!)$ is indeed generated by $0\ (P_i \bullet \rho)\ 1$. □

[8] By virtue of theorem 2.1.3°2, all right hand sides are defined since $P_i \bullet \rho \neq \rightleftarrows_j$ for $b_j = \nearrow$.

**Definition 2.2.1°3.** We define the prism functors from definition 1.2.2°2 on jet setoids:

- The carrier is still $X \times \{0, 1\}$,
- We set $(x, u) \frown_{\mathrm{eq}} (y, v)$ if $x \frown_{\mathrm{eq}} y$ and $u = v$, i.e. the underlying setoid is $X \times \llparenthesis \frown_\top \rrparenthesis$.
- Jet relations are generated as in definition 1.2.2°2.

**Definition 2.2.1°4.** We adapt definition 1.4.4°1 of the separated product to jet setoids by taking the cartesian product of the underlying setoids and replacing equality with equivalence in definition 1.4.4°1.

**Lemma 2.2.1°5.** Let $\mu : \vec{a} \to \vec{b}$ be a polarized reshuffle. Then $\mathsf{JetSetoid}(\mu)$, seen as a non-enriched functor, preserves colimits.

*Proof.* Take a diagram $D : \mathcal{S} \to \mathsf{JetSetoid}(\vec{a})$. We prove (assuming propositional extensionality) that

$$\mu \operatorname{colim} D = \operatorname{colim}(\mu \circ D).$$

- The functor $\mathsf{JetSetoid}(\mu)$ does not modify the carrier, and the carrier of a colimit of jet setoids is the colimit of the carriers. Write $\iota_s$ for the functions $D(s) \to \operatorname{colim} D$.
- One checks that, on both sides, elements $x$ and $y$ are equivalent if and only if there exist $x = x_0, x_1, x_2, \ldots, x_n = y$ such that for each $0 \leq i < n$, there exists an $s \in \mathrm{Obj}(\mathcal{S})$ and $u, v \in D(s)$ such that $x_i = \iota_s(u)$ and $x_{i+1} = \iota_s(v)$ and $u\ (\frown_{\mathrm{eq}} \bullet \mu)\ v$.
- One checks that, on both sides, $x \rightarrowtriangle_i y$ holds in $\mu \operatorname{colim} D$ if and only if there exist an $s \in \mathrm{Obj}(\mathcal{S})$, $x \frown_{\mathrm{eq}}^{\mu \operatorname{colim} D} \iota_s(x')$ and $y \frown_{\mathrm{eq}}^{\mu \operatorname{colim} D} \iota_s(y')$ such that $x'\ (\rightarrowtriangle_i \cdot \mu)\ y'$ in $D(s)$. $\square$

**Proposition 2.2.1°6.** Let $\mu : \vec{a} \to \vec{b}$ be a polarized reshuffle with right adjoint $\mu \dashv \rho$. Then the action of $\mathsf{JetSetoid}(\mu)$ on prisms can be looked up in the following table:[9,10]

| $i \cdot \rho$ | $\odot_{\mathrm{eq}}$ | $\otimes_j$ | $\oplus_j$ | $\ominus_j$ | $\odot_\top$ |
|---|---|---|---|---|---|
| $\mu(X \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis)$ | $(\mu X) \times \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis$ | $(\mu X) * \llparenthesis \leftrightarrowtriangle_j \rrparenthesis$ | $(\mu X) \ltimes \llparenthesis \rightarrowtriangle_j \rrparenthesis$ | $(\mathsf{Op}_j \mu X) \ltimes \llparenthesis \leftarrowtriangle_j \rrparenthesis$ | $(\mu X) \times \llparenthesis \frown_\top \rrparenthesis$ |
| $\mu(X \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis)$ | $(\mu X) \times \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis$ | $(\mu X) * \llparenthesis \leftrightarrowtriangle_j \rrparenthesis$ | $(\mu X) \ltimes \llparenthesis \leftarrowtriangle_j \rrparenthesis$ | $(\mathsf{Op}_j \mu X) \ltimes \llparenthesis \rightarrowtriangle_j \rrparenthesis$ | $(\mu X) \times \llparenthesis \frown_\top \rrparenthesis$ |

Note that a cartesian product with $\llparenthesis \frown_{\mathrm{eq}} \rrparenthesis$ or $\llparenthesis \frown_\top \rrparenthesis$ is the same as a separated product, since neither jet setoid has any jets between non-equivalent elements.

*Proof.* We mainly prove the upper row of the table For the lower row, we use that $X \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis = \mathsf{Op}_i(X \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis)$ so the result can be derived from the upper row if we can rewrite $\mu \circ \mathsf{Op}_i$.

Each time, it is clear that the carrier is just $X \times \{0, 1\}$. We can write $X \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis$ as the colimit of $\mathsf{Op}_i(X)$, $X$, a copy of $\llparenthesis P_i \rrparenthesis$ for every element of $X$, and a copy of the terminal jet setoid for every element of $X \times \{0, 1\}$. By lemma 2.2.1°5, $\mathsf{JetSetoid}(\mu)$ preserves that colimit. By proposition 2.2.1°2, $\mu \llparenthesis \rightarrowtriangle_i \rrparenthesis = \llparenthesis \rightarrowtriangle_i \bullet \rho \rrparenthesis$. Thus, we see that $\mu(X \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis)$ is a colimit of $\mu \mathsf{Op}_i X$, $\mu X$ and copies of $\llparenthesis \rightarrowtriangle_i \bullet \rho \rrparenthesis$. We case distinguish on $i \cdot \rho$.

$\boxed{\odot_{\mathrm{eq}}}$ If $i \cdot \rho = \odot_{\mathrm{eq}}$, then $\mathsf{Op}_i \circ \rho = \rho$. Then by uniqueness of the adjoint, $\mu \circ \mathsf{Op}_i = \mu$. Then the colimit produces $(\mu X) * \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis$, and the result for the opposite prism is the same.

$\boxed{\otimes_j}$ If $i \cdot \rho = \otimes_j$, then $\mathsf{Op}_i \circ \rho = \rho$. Then by uniqueness of the adjoint, $\mu \circ \mathsf{Op}_i = \mu$. Then the colimit produces $(\mu X) * \llparenthesis \leftrightarrowtriangle_j \rrparenthesis$, and the result for the opposite prism is the same.

$\boxed{\oplus_j}$ If $i \cdot \rho = \oplus_j$, then by proposition 2.1.3°11, $j \cdot \mu = \oplus_i$ and $\mu \circ \mathsf{Op}_i = \mathsf{Op}_j \circ \mu$. This means that we are taking the $j$-opposite at the original source-side of the twisted prism, which is also the new source-side, so we get a $j$-twisted prism $(\mu X) \ltimes \llparenthesis \rightarrowtriangle_j \rrparenthesis$. The image of the $i$-opposite is then the $j$-opposite of the image, which is the opposite prism $(\mu X) \ltimes \llparenthesis \leftarrowtriangle_j \rrparenthesis$.

[9] Note that if $b_j = \bigcirc$, then the results for $\odot_j$ are consistent.

[10] Note that if $i \cdot \rho = \circledast_j$, then by eq. (2.7), $b_j = \bigcirc$.

$\boxed{\ominus_j}$ If $i \cdot \rho = \ominus_j$, then by proposition 2.1.3°11, $j \cdot \mu = \ominus_i$ and $\mu \circ \mathsf{Op}_i = \mathsf{Op}_j \circ \mu$. This means that we are taking the $j$-opposite at the original source-side of the twisted prism, which is the new *target*-side. Equivalently, we can take the $j$-opposite on both sides and then again at the source-side, so we get a $j$-twisted prism $(\mathsf{Op}_j \mu X) \ltimes (\!|\mapsto_j|\!)$. The image of the $i$-opposite is then the $j$-opposite of the image, which is the opposite prism $(\mathsf{Op}_j \mu X) \ltimes (\!|\leftarrow_j|\!)$.

$\boxed{\odot_\top}$ If $i \cdot \rho = \odot_\top$, then $\mathsf{Op}_i \circ \rho = \rho$. Then by uniqueness of the adjoint, $\mu \circ \mathsf{Op}_i = \mu$. Then the colimit produces $(\mu X) * (\!|\frown_\top|\!) = (\mu X) \times (\!|\frown_\top|\!)$, and the result for the opposite prism is the same. □

## 2.3 Semantical Notations

### 2.3.1 Presheaf Notations

We use the presheaf notations from [Nuy18]. Concretely:

- The application of a presheaf $\Gamma \in \widehat{\mathcal{W}}$ to an object $W \in \mathcal{W}$ is denoted $W \Rightarrow \Gamma$.
- The restriction of $\gamma : W \Rightarrow \Gamma$ by $\varphi : V \to W$ is denoted $\gamma \circ \varphi$ or $\gamma\varphi$.
- The application of a presheaf morphism $\sigma : \Gamma \to \Delta$ to $\gamma : W \Rightarrow \Gamma$ is denoted $\sigma \circ \gamma$ or $\sigma\gamma$.
  - By naturality of $\sigma$, we have $\sigma \circ (\gamma \circ \varphi) = (\sigma \circ \gamma) \circ \varphi$.
- If $\Gamma \in \widehat{\mathcal{W}}$ and $T \in \mathrm{Ty}(\Gamma)$ (also denoted $\Gamma \vdash T\,\mathsf{type}$), i.e. $T$ is a presheaf over the category of elements $\mathcal{W}/\Gamma$, then we write the application of $T$ to $(W, \gamma)$ as $(W \Vdash T[\gamma\rangle)$ and $t \in (W \Vdash T[\gamma\rangle)$ as $W \Vdash t : T[\gamma\rangle$.
  - By definition of type substitution in a presheaf CwF, we have $(W \Vdash T[\sigma][\gamma\rangle) = (W \Vdash T[\sigma\gamma\rangle)$
- The restriction of $W \Vdash t : T[\gamma\rangle$ by $\varphi : (V, \gamma \circ \varphi) \to (W, \gamma)$ is denoted as $W \Vdash t\langle\varphi\rangle : T[\gamma\varphi\rangle$.
- If $t \in \mathrm{Tm}(\Gamma, T)$ (also denoted $\Gamma \vdash t : T$), then the application of $t$ to $(W, \gamma)$ is denoted $V \Vdash t[\gamma\rangle : T[\gamma\rangle$.
  - The naturality condition for terms is then expressed as $t[\gamma\rangle\langle\varphi\rangle = t[\gamma\varphi\rangle$.
  - By definition of term substitution in a presheaf CwF, we have $t[\sigma][\gamma\rangle = t[\sigma\gamma\rangle$.
- We omit applications of the isomorphisms $(W \Rightarrow \Gamma) \cong (\mathbf{y}W \to \Gamma)$ and $(W \Vdash T[\gamma\rangle) \cong (\mathbf{y}W \vdash T[\gamma])$. This is not unreasonable: e.g. given $W \Vdash t : T[\gamma\rangle$, the term $\mathbf{y}W \vdash t' : T[\gamma]$ is defined by $t'[\varphi\rangle := t\langle\varphi\rangle$.

One advantage of these notations is that we can put presheaf cells in diagrams; we will use double arrows when doing so.

### 2.3.2 End and Co-ends

We use existential quantifiers over objects of a category to denote ends and co-ends, i.e. if $F : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}$, then its co-end is denoted $\exists X.F(X, X)$ and its end is denoted $\forall X.F(X, X)$.

## 2.4 Semantics on Jet Topes and Jet Topic Sets

In this section, we investigate how polarized reshuffles act on jet topes and on presheaves over these, which we call jet topic sets.

### 2.4.1 Semantics on Jet Topes

In this section, we want to construct a 2-functor that sends polarized reshuffles $\mu : \vec{a} \to \vec{b}$ to functors $\mathsf{JetTope}(\vec{a}) \to \mathsf{JetTope}(\vec{b})$, thus giving our mode theory of polarized reshuffles a semantics on jet topes.

However, it is pointless to distinguish between two modalities when they have the exact same semantics, and it is at least roughly pointless to do so when they have roughly the same semantics. Looking at definition 1.3.0°2 of jet topes, we see that:

- Due to $i$-antisymmetry at directed degrees, we know that the $i$-equijet relation is equivalent to the $(i-1)$-infrajet relation (if $i > 0$) or to equality (if $i = 0$). This means that a modality $\mu$ will have the same semantics, regardless of whether $j \cdot \mu = \otimes_i$ or $j \cdot \mu = \circledast_{i-1}$ (if $i > 0$) or $j \cdot \mu = \odot_{\mathrm{eq}}$ (if $i = 0$). For this reason, we will disallow any mentions of $\otimes_i$ when $a_i = \nearrow$.
- Due to $(n-1)$-connectedness, the $(n-1)$-infrajet relation is roughly equivalent to the 'true' relation. For this reason, we will disallow any mentions of $\odot_\top$, as $\circledast_{n-1}$ can be mentioned instead.

Taking these two requirements together, by theorem 2.1.3°2, we get exactly the statement that $\mu$ must have a right adjoint.

**Theorem 2.4.1°1.** Let $\mu \in \mathrm{Hom}_{\mathsf{PoResh}}(\vec{a}, \vec{b})$ have a right adjoint $\mu \dashv \rho$. Then the functor $\mu : \mathsf{JetSet}(\vec{a}) \to \mathsf{JetSet}(\vec{b})$ sends jet topes to jet topes if and only if $\rho$ has a further right adjoint $\mu \dashv \rho \dashv \tau$.

By consequence, there is a pseudofunctor $\mathsf{JetTope} : \mathsf{PoResh}^{\bullet\circ\circ} \to \mathsf{Cat}$ sending

- an object $\vec{a}$ to $\mathsf{JetTope}(\vec{a})$,
- a morphism $\mu : \vec{a} \to \vec{b}$ to the restriction of $\mathsf{JetSet}(\mu) : \mathsf{JetSet}(\vec{a}) \to \mathsf{JetSet}(\vec{b})$ to $\mathsf{JetTope}(\vec{a})$,
- a 2-cell $\alpha : \mu \to \nu : \vec{a} \to \vec{b}$ to the restriction of $\mathsf{JetSet}(\alpha)$ to $\mathsf{JetTope}(\vec{a})$.

*Proof.* We first prove the 'only if' direction. Note that, by theorem 2.1.3°5, the property that $\rho$ has a further right adjoint is decidable, so we can do this from the absurd. Assume that $\rho$ does not have a right adjoint.

- Either $\rho$ violates eq. (2.8), i.e. $\downarrow(\top) \cdot \rho = \odot_\top$. Then by proposition 2.2.1°2, $\mu\llparenthesis \rightarrowtriangle_{\downarrow(\top)} \rrparenthesis = \llparenthesis \frown_\top \rrparenthesis$, where $\llparenthesis \rightarrowtriangle_{\downarrow(\top)} \rrparenthesis$ is a jet tope but $\llparenthesis \frown_\top \rrparenthesis$ is not, as it is not $(n-1)$-connected (if $n > 0$) or not a subsingleton (if $n = 0$).
- Or $\rho : \vec{b} \to \vec{a}$ violates eq. (2.9), i.e. there is some $i \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a})$ with $i \cdot \rho = \otimes_j$ and $b_j = \nearrow$. Then by proposition 2.2.1°2, $\mu\llparenthesis \rightarrowtriangle_i \rrparenthesis = \llparenthesis \leftrightarrowtriangle_j \rrparenthesis$, where $\llparenthesis \rightarrowtriangle_i \rrparenthesis$ is a jet tope but $\llparenthesis \leftrightarrowtriangle_j \rrparenthesis$ is not, as it is not $j$-antisymmetric.

Now we prove the 'if' direction. Assume that $\rho$ does have a further right adjoint $\mu \dashv \rho \dashv \tau$ and take a jet tope $X$. We prove that $\mu X$ is a jet tope (definition 1.3.0°2). Informed by the construction in definition 2.2.0°8, we will regard both $X$ and $\mu X$ as jet *setoids*.

1. Clearly, $\mathbf{Q}(\mu X)$ is finite, as $\mathbf{Q}(X)$ was finite.
2. To see that $\mu X$ is $\downarrow(\top)$-connected (i.e. $(n-1)$-connected if $n > 0$, or eq-connected if $n = 0$), note that $x \rightarrowtriangle_{\downarrow(\top)} y$ in $\mu X$ if and only if $x \rightleftarrows^*_{\mathrm{eq}\cdot\mu} x' \rightarrowtriangle_{\downarrow(\top)\cdot\mu} y' \rightleftarrows^*_{\mathrm{eq}\cdot\mu} y$, and in particular if $x \rightarrowtriangle_{\downarrow(\top)\cdot\mu} y$. Now $\downarrow(\top) \cdot \mu > \otimes_{\downarrow(\top)}$ by eq. (2.12). So $\rightleftarrows_{\downarrow(\top)} \bullet \mu = \rightleftarrows_{\downarrow(\top)}$. Thus, since $\rightleftarrows^*_{\downarrow(\top)}$ is always true in $X$, we conclude that $\rightleftarrows^*_{\downarrow(\top)}$ is always true in $\mu X$.
3. We show that $\mu X$ is $j$-orientable for every $j \in \mathsf{Deg}(\vec{b})$. Let $x \rightleftarrows^*_j y$ in $\mu X$. Then
$$x\ ((\frown_{\mathrm{eq}} \bullet \mu)^* \otimes (\rightleftarrows_j \bullet \mu) \otimes (\frown_{\mathrm{eq}} \bullet \mu)^*)^*\ y$$
in $X$. Now $\mathrm{eq} < j$ so $\mathrm{eq} \cdot \mu \ll j \cdot \mu$, so $\frown_{\mathrm{eq}} \bullet \mu \leq \leftrightarrowtriangle_j \bullet \mu \leq \rightleftarrows_j \bullet \mu$, so we can conclude more simply that $x\ (\rightleftarrows_j \bullet \mu)^*\ y$. By case analysis on $\mathsf{pol}(j \cdot \mu)$, where the symmetric cases are trivial, and using that $X$ is $\mathsf{deg}(j \cdot \mu)$-orientable, we can conclude that $x(\rightarrowtriangle_j \bullet\mu)^* y$ or $x(\leftarrowtriangle_j \bullet\mu)^* y$ in $X$, and therefore that $x \rightarrowtriangle^*_j y$ or $x \leftarrowtriangle^*_j y$ in $\mu X$.
4. We show that $\mu X$ is $j$-acyclic for every $j \in \mathsf{Deg}(\vec{b})$ by similar reasoning.
5. Assume that $b_j = \nearrow$. We show that $\mu X$ is $j$-antisymmetric, i.e. that $x \leftrightarrowtriangle^{\mu X}_j y$ implies $x \rightleftarrows^{\mu X}_{\downarrow(j)} y$. Let $p'_{i'} = \downarrow(j) \cdot \mu$ and $p_i = j \cdot \mu$. We case split on how $i'$ compares to $i$

   $i' = i$ Then without relying on $X$ being a jet tope, we can prove $j$-antisymmetry by case distinction on $p'$ and $p$.

   $i' = \downarrow(i)$ Then by eqs. (2.9) and (2.13), we find that $p \in \{\oplus, \ominus\}$. By eq. (2.9), $p' \in \{\oplus, \ominus, \circledast\}$. Thus, we see that $\rightleftarrows_{\downarrow(j)} \bullet \mu = \rightleftarrows_{\downarrow(i)}$ and $\leftrightarrowtriangle_j \bullet \mu = \leftrightarrowtriangle_i$. Then we can invoke $i$-antisymmetry of $X$.

   $i' < \downarrow(i)$ Then we have $\mathsf{deg}(\downarrow(j) \cdot \mu) < \downarrow(i)$ and $j \cdot \mu \geq \circledast_{\downarrow(i)}$, so by eq. (2.13), this is impossible as $b_j = \nearrow$. ∎

### 2.4.2 Adjoint Triple Semantics on Jet Topic Sets

**Definition 2.4.2°1.** The category of $\vec{a}$**-jet topic sets** is the presheaf category $\mathsf{JTSet}(\vec{a}) := \mathsf{Psh}(\mathsf{JetTope}(\vec{a}))$.

**Theorem 2.4.2°2.** We have the following pseudofunctors:

- $\mathsf{JTSet}^{\bullet\circ\circ} : \mathsf{PoResh}^{\bullet\circ\circ} \to \mathsf{Cat}^{\bullet\circ\circ}$,
- $\mathsf{JTSet}^{\circ\bullet\circ} : \mathsf{PoResh}^{\circ\bullet\circ} \to \mathsf{Cat}^{\circ\bullet\circ}$,
- $\mathsf{JTSet}^{\circ\circ\bullet} : \mathsf{PoResh}^{\circ\circ\bullet} \to \mathsf{Cat}^{\circ\circ\bullet}$,

to the 2-category of categories, functors with a specified number of left/right adjoints (notation 2.1.3°1), and natural transformations, where:

- each functor sends an object $\vec{a}$ to $\mathsf{Psh}(\mathsf{JetTope}(\vec{a}))$,
- a triple of polarized reshuffles $\mu \dashv \rho \dashv \tau$ is sent by the respective functors to
$$\mathsf{JTSet}^{\bullet\circ\circ}(\mu) = \mu_! \qquad \dashv \qquad \mathsf{JTSet}^{\circ\bullet\circ}(\rho) = \mu^* \qquad \dashv \qquad \mathsf{JTSet}^{\circ\circ\bullet}(\tau) = \mu_*,$$
of which $\mu^*$ is a strict CwF morphism [Dyb95] and $\mu_*$ is a weak CwF morphism [Nuy18, def. 2.1.1][Nuy20a, def. 3.2.5] w.r.t. the standard CwF structure on a presheaf category [Hof97, ch. 4][Nuy20a, §4.1].
- 2-cells are also respected.

*Proof.* In general, a functor $F : \mathcal{V} \to \mathcal{W}$ gives rise to a triple of adjoint functors $F_! \dashv F^* \dashv F_*$ where $F^* : \mathsf{Psh}(\mathcal{W}) \to \mathsf{Psh}(\mathcal{V})$ is precomposition with $F$ and the other two are left and right Kan extension along $F$ [Sta, nLac][Nuy20a, §2.3.8]. It is in general true that $F^*$ is a strict CwF morphism and $F_*$ is a weak one [Nuy18, thm. 2.2.1 & prop. 2.2.9][Nuy20a, §6.4].

The middle operation amounts to a strict 2-functor $\mathsf{Cat}^{\mathsf{co;op}} \to \mathsf{Cat}$, so the other operations amount to pseudofunctors $\mathsf{Cat} \to \mathsf{Cat}$. We compose these operations with the pseudofunctorial semantics of polarized reshuffles on jet topes (theorem 2.4.1°1), and finally use the fact that $\mathsf{PoResh}^{\circ\circ\bullet} \cong (\mathsf{PoResh}^{\circ\bullet\circ})^{\mathsf{co;op}} \cong \mathsf{PoResh}^{\bullet\circ\circ}$, with each isomorphism sending a polarized reshuffle to its left adjoint. $\square$

### 2.4.3 No Adjoint Pair Semantics on Jet Topic Sets

Theorem 2.4.2°2 is a fairly good result: any polarized reshuffle that is part of a chain of at least three adjoints, now has semantics on jet topic sets. Moreover, if we have a chain of four adjoints $\kappa \dashv \mu \dashv \rho \dashv \tau$, then by uniqueness of the adjoint, we find that $\mathsf{JTSet}^{\circ\bullet\circ}(\mu) \cong \mathsf{JTSet}^{\bullet\circ\circ}(\mu)$ and similar for $\rho$. So the 3 pseudofunctors essentially agree on the intersection of their domains.

However, we can have commutative diagrams in $\mathsf{PoResh}$ each of whose sides has semantics in jet topic sets, but where the composites of the sides do not have such semantics, so that commutativity is not necessarily preserved. Consider, for example, the following diagram with polarized reshuffles from example 2.1.3°8:

$$\begin{tikzcd}
{[\nearrow]} \arrow[r, "\sqcap_0 = \langle \circledast_0 \mid \rangle"] \arrow[d, "\nabla_1 = \langle \odot_{\mathrm{eq}} \mid \oplus_0, \odot_\top \rangle"'] & {[]} \arrow[d, "\nabla_0 = \langle \odot_{\mathrm{eq}} \mid \odot_\top \rangle"] \\
{[\nearrow, \nearrow]} \arrow[r, "\sqcap_0 = \langle \circledast_0 \mid \oplus_1 \rangle"'] & {[\nearrow]}
\end{tikzcd}$$

The horizontal sides are in $\mathsf{PoResh}^{\bullet\circ\circ}$, since $\sqcap_0 \dashv \triangle_0 \dashv \sqcup_0$. The vertical sides are in $\mathsf{PoResh}^{\circ\circ\bullet}$, since $\triangle_i \dashv \sqcup_i \dashv \nabla_i$. The diagonals compose to $\langle \circledast_0 \mid \odot_\top \rangle : [\nearrow] \to [\nearrow]$, which by theorem 2.1.3°2 has neither a left, nor a right adjoint. So each of the sides has semantics on jet topic sets, but we currently do not know whether the diagram still commutes there. We cannot invoke the composition law of any pseudofunctor provided by theorem 2.4.2°2, because the composition operands all require a different pseudofunctor.

Unfortunately, we have the following **no-go theorem**:

**Theorem 2.4.3°1.** The following, equivalent[11], objects **do not exist**:

- A pseudofunctor $\mathsf{JTSet}^{\bullet\circ} : \mathsf{PoResh}^{\bullet\circ} \to \mathsf{Cat}^{\bullet\circ}$ that extends both $\mathsf{JTSet}^{\bullet\circ\circ}$ and $\mathsf{JTSet}^{\circ\bullet\circ}$ up to natural isomorphism,
- A pseudofunctor $\mathsf{JTSet}^{\circ\bullet} : \mathsf{PoResh}^{\circ\bullet} \to \mathsf{Cat}^{\circ\bullet}$ that extends both $\mathsf{JTSet}^{\circ\bullet\circ}$ and $\mathsf{JTSet}^{\circ\circ\bullet}$ up to natural isomorphism.

*Proof.* Assume the former exists, and consider the polarized reshuffles

$$\sqcup_0 = \langle \odot_{\mathrm{eq}} \,|\, \oplus_1 \rangle : [\nearrow, \nearrow] \to [\nearrow], \qquad \triangle_1 = \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \circledast_0 \rangle : [\nearrow] \to [\nearrow, \nearrow]$$

from example 2.1.3°8, which compose to $\sqcup_0 \triangle_1 = \langle \odot_{\mathrm{eq}} \,|\, \circledast_0 \rangle : [\nearrow] \to [\nearrow]$. This latter reshuffle is easily computed to be idempotent, i.e. $\langle \odot_{\mathrm{eq}} \,|\, \circledast_0 \rangle^2 = \langle \odot_{\mathrm{eq}} \,|\, \circledast_0 \rangle$. Then by pseudofunctoriality, we expect a natural isomorphism $\mathsf{JTSet}^{\bullet\circ}(\sqcup_0 \triangle_1)^2 \cong \mathsf{JTSet}^{\bullet\circ}(\sqcup_0 \triangle_1)$, but we will disprove this.

By pseudofunctoriality, we know that

$$\mathsf{JTSet}^{\bullet\circ}(\sqcup_0 \triangle_1) \cong \mathsf{JTSet}^{\bullet\circ}(\sqcup_0) \circ \mathsf{JTSet}^{\bullet\circ}(\triangle_1) \cong \mathsf{JTSet}^{\circ\bullet\circ}(\sqcup_0) \circ \mathsf{JTSet}^{\bullet\circ\circ}(\triangle_1) = (\triangle_0)^* \circ (\triangle_1)_!,$$

where $\triangle_0 = \langle \odot_{\mathrm{eq}} \,|\, \odot_{\mathrm{eq}}, \oplus_0 \rangle : [\nearrow] \to [\nearrow, \nearrow]$. Consider the jet topic set $\mathbf{y}\llparenthesis \rightarrowtriangle_0 \rrparenthesis \in \mathsf{JTSet}([\nearrow])$, which is the Yoneda-embedding of the jet tope $\llparenthesis \rightarrowtriangle_0 \rrparenthesis \in \mathsf{JetTope}([\nearrow])$. We compute the sets of 0-jets of its images under either the above functor or its square:

$$\begin{array}{rll}
& \llparenthesis \rightarrowtriangle_0 \rrparenthesis \Rightarrow (\triangle_0)^* (\triangle_1)_! \mathbf{y} \llparenthesis \rightarrowtriangle_0 \rrparenthesis & \\
= & \triangle_0 \llparenthesis \rightarrowtriangle_0 \rrparenthesis \Rightarrow (\triangle_1)_! \mathbf{y} \llparenthesis \rightarrowtriangle_0 \rrparenthesis & \\
\cong & \triangle_0 \llparenthesis \rightarrowtriangle_0 \rrparenthesis \Rightarrow \mathbf{y} \triangle_1 \llparenthesis \rightarrowtriangle_0 \rrparenthesis & (F_! \circ \mathbf{y} \cong \mathbf{y} \circ F) \\
= & \llparenthesis \rightarrowtriangle_0 \bullet \sqcup_0 \rrparenthesis \Rightarrow \mathbf{y} \llparenthesis \rightarrowtriangle_0 \bullet \sqcup_1 \rrparenthesis & (\text{proposition 2.2.1°2}) \\
= & \llparenthesis \rightarrowtriangle_1 \rrparenthesis \Rightarrow \mathbf{y} \llparenthesis \rightarrowtriangle_0 \rrparenthesis & \\
= & \llparenthesis \rightarrowtriangle_1 \rrparenthesis \to \llparenthesis \rightarrowtriangle_0 \rrparenthesis. &
\end{array}$$

Note that, after applying the reshuffles to the jet topes, we obtain jet topes in $\mathsf{JetTope}([\nearrow, \nearrow])$. Since any two nodes in $\llparenthesis \rightarrowtriangle_0 \rrparenthesis$ are connected by a 1-equijet, any function $\{0, 1\} \to \{0, 1\}$ between the carriers is a jet set (hence jet tope) morphism.

On the other hand, we have

$$\begin{array}{rll}
& \llparenthesis \rightarrowtriangle_0 \rrparenthesis \Rightarrow (\triangle_0)^* (\triangle_1)_! (\triangle_0)^* (\triangle_1)_! \mathbf{y} \llparenthesis \rightarrowtriangle_0 \rrparenthesis & \\
\cong & \llparenthesis \rightarrowtriangle_1 \rrparenthesis \Rightarrow (\triangle_1)_! (\triangle_0)^* \mathbf{y} \llparenthesis \rightarrowtriangle_0 \rrparenthesis & (\text{by similar reasoning as above}) \\
= & \exists X^{\in \mathsf{JetTope}([\nearrow])}. (\llparenthesis \rightarrowtriangle_1 \rrparenthesis \to \triangle_1 X) \times (X \Rightarrow (\triangle_0)^* \mathbf{y} \llparenthesis \rightarrowtriangle_0 \rrparenthesis) & (\text{construction of left Kan extension}) \\
= & \exists X^{\in \mathsf{JetTope}([\nearrow])}. (\llparenthesis \rightarrowtriangle_1 \rrparenthesis \to \triangle_1 X) \times (\triangle_0 X \to \llparenthesis \rightarrowtriangle_0 \rrparenthesis). &
\end{array}$$

(Note that $\exists$ denotes a co-end (section 2.3.2).) Now a 1-jet in $\triangle_1 X = \langle \odot_{\mathrm{eq}} \,|\, \oplus_0, \circledast_0 \rangle X$ is a 1-infrajet in $X$. Such an infrajet either points forwards, in which case $f : \llparenthesis \rightarrowtriangle_1 \rrparenthesis \to \triangle_1 X$ factors over some $\llparenthesis \rightarrowtriangle_1 \rrparenthesis \to \triangle_1 \llparenthesis \rightarrowtriangle_0 \rrparenthesis$, or it points backwards, in which case it factors over $\llparenthesis \rightarrowtriangle_1 \rrparenthesis \to \triangle_1 \llparenthesis \leftarrowtriangle_0 \rrparenthesis$. If it factors over both, since $X$ is a jet tope, by 0-antisymmetry, it factors over $\llparenthesis \rightarrowtriangle_1 \rrparenthesis \to \triangle_1 \top$. Thus, the result of the computation above is the following pushout

$$\begin{array}{rll}
\ldots \cong & (\triangle_0 \llparenthesis \rightarrowtriangle_0 \rrparenthesis \to \llparenthesis \rightarrowtriangle_0 \rrparenthesis) \uplus_{(\triangle_0 \top \to \llparenthesis \rightarrowtriangle_0 \rrparenthesis)} (\triangle_0 \llparenthesis \leftarrowtriangle_0 \rrparenthesis \to \llparenthesis \rightarrowtriangle_0 \rrparenthesis) & \\
= & (\llparenthesis \rightarrowtriangle_1 \rrparenthesis \to \llparenthesis \rightarrowtriangle_0 \rrparenthesis) \uplus_{(\top \to \llparenthesis \rightarrowtriangle_0 \rrparenthesis)} (\llparenthesis \leftarrowtriangle_1 \rrparenthesis \to \llparenthesis \rightarrowtriangle_0 \rrparenthesis) & (\text{proposition 2.2.1°2}) \\
\cong & (\{0, 1\} \to \{0, 1\}) \uplus_{\{0,1\}} (\{0, 1\} \to \{0, 1\}) &
\end{array}$$

so all non-constant functions between the carriers now occur twice! The intuitive reason is in fact simple: a 0-jet in $\mathsf{JTSet}^{\bullet\circ}(\langle \odot_{\mathrm{eq}} \,|\, \circledast_0 \rangle)\Gamma$ should be a 0-jet in $\Gamma$ pointing either way, i.e. unless the jet is constant, we make a choice in which direction it will point, and then we pick a jet in that direction. But then when

[11] Assuming an adjoint can be picked if one exists.

we apply this operation twice, a 0-jet in $\mathsf{JTSet}^{\bullet\circ}(\langle \odot_{\mathrm{eq}} \,|\, \circledast_0 \rangle)\mathsf{JTSet}^{\bullet\circ}(\langle \odot_{\mathrm{eq}} \,|\, \circledast_0 \rangle)\Gamma$ should be a 0-jet in $\Gamma$ pointing *either either* way, i.e. unless the jet is constant, we choose two bits of which one decides in which direction it will point, and then we pick a jet in that direction. Note the choice of an unused bit.

Now assume $\mathsf{JTSet}^{\circ\bullet}$ exists. The right adjoint to $\sqcup_0 \triangle_1$ is $\sqcup_1 \triangledown_0 = \langle \odot_{\mathrm{eq}} \,|\, \otimes_0 \rangle$, which is necessarily also idempotent. By similar reasoning as above, we will have

$$\mathsf{JTSet}^{\circ\bullet}(\sqcup_1 \triangledown_0) = (\triangle_1)^* \circ (\triangle_0)_*,$$

which is right adjoint to $(\triangle_0)^* \circ (\triangle_1)_!$ and therefore also non-idempotent. □

### 2.4.4 Conclusion

In conclusion, if we intend to model Naturality Pretype Theory in jet topic sets, then we will confine ourselves to using $\mathsf{PoResh}^{\circ\bullet\circ}$ as our mode theory. This means that our internal modalities will be indexed by polarized reshuffles which have a left and a right adjoint, i.e. satisfy eqs. (2.6) to (2.9). It is almost certainly possible to drop eq. (2.8) so as to include modalities for irrelevance and shape-irrelevance [ND18], but we leave this to potential future work.

Thus, the *locks* in our MTT instance will then correspond to polarized reshuffles in $\mathsf{PoResh}^{\bullet\circ\circ}$, i.e. those satisfying eqs. (2.8), (2.9), (2.12) and (2.13); where the future work hinted-at above might allow dropping eq. (2.12).

## 2.5 Semantics on Jet Cubes and Jet Cubical Sets

In this section, we investigate how polarized reshuffles act on jet topes and on presheaves over these, which we call jet topic sets.

### 2.5.1 Semantics on Jet Cubes and Jet Cuboids

In this section, we want to construct a 2-functor that sends polarized reshuffles $\mu : \vec{a} \to \vec{b}$ to functors $\mathsf{JetCube}^{\square}_M(\mathsf{fb}, \vec{a}) \to \mathsf{JetCube}^{\square}_M(\mathsf{fb}, \vec{b})$, with $M \in \{\mathsf{IPt}_2, \mathsf{Boo}\}$, thus giving our mode theory of polarized reshuffles a semantics on fb-jet-cubes. Recall that these are equivalent to f-jet-cubes (proposition 1.4.2°10).

Since jet cubes are jet topes (corollary 1.4.2°9), the reasoning at the start of section 2.4.1 applies here as well, so that we will only consider modalities $\mu$ that have a right adjoint $\mu \dashv \rho$. Since jet cubes are realized as jet sets by repeated application of twisted prism functors to the terminal jet set (definition 1.4.2°5), we can deduce the action of $\mu$ on jet cubes from its action on twisted prisms, which is given by proposition 2.2.1°6. We see that the outcome may be either: again a twisted prism, or a separated/cartesian product with $\llparenthesis \frown_{\mathrm{eq}} \rrparenthesis$ (i.e. the terminal jet set), $\llparenthesis \leftrightarrow_j \rrparenthesis$ or $\llparenthesis \frown_\top \rrparenthesis$ (i.e. a discrete jet set with two elements). We have to rule out $\llparenthesis \leftrightarrow_j \rrparenthesis$ and $\llparenthesis \frown_\top \rrparenthesis$, since these cannot be realized by jet cubes, but we can support $\llparenthesis \frown_{\mathrm{eq}} \rrparenthesis$ since $\sqcup \times \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis \cong \mathrm{Id}$. If we look at what this entails for $\rho$, we see that $\rho$ cannot mention $\otimes_j$ for directed $j$, nor $\odot_\top$, i.e. (by theorem 2.1.3°2) $\rho$ has to have a right adjoint.

Analogous to what we did in section 2.2, we will first work on a variant of jet cubes, called jet cuboids[12] $V$, which can have variables typed with the 'equality interval' $\llparenthesis \frown_{\mathrm{eq}} \rrparenthesis$, and whose jet endpoint model $\mathsf{JEP}(V)$ is not a jet set but a jet setoid. This allows us to work in a strictly 2-functorial setting longer.

#### 2.5.1 (a) Cuboids and Jet Cuboids

**Definition 2.5.1°1.** Given a monad $M$ and $\text{⌑} \in \{\boxslash, \square\}$, we define the **setoid-enriched category of cuboids** $\mathsf{Cuboid}^{\text{⌑}}_M$, with strict 2-functors $\mathbf{U} : \mathbf{U}_1\mathsf{Cuboid}^{\text{⌑}}_M \to \mathsf{Cube}^{\text{⌑}}_M$ and $\mathbf{Q} : \mathbf{Q}_1\mathsf{Cuboid}^{\text{⌑}}_M \to \mathsf{Cube}^{\text{⌑}}_M$ as the category:

- whose objects are lists of elements of $\{\mathbb{I}, \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis\}$, denoted in the style $(\mathbf{i}_0 : \mathbb{I}, \ldots, \mathbf{i}_{n-1} : \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis)$ where the $\mathbf{i}_k$ are interpreted as de Bruijn indices, and where

$$\mathbf{U}() = (), \qquad \mathbf{U}(W, \mathbf{i} : \mathbb{I}) = (\mathbf{U}W, \mathbf{i} : \mathbb{I}), \qquad \mathbf{U}(W, \mathbf{i} : \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis) = (\mathbf{U}W, \mathbf{i} : \mathbb{I}),$$

[12] The -oid is in the sense of setoid here.

$$\mathbf{Q}() = (), \qquad \mathbf{Q}(W, \mathbf{i} : \mathbb{I}) = (\mathbf{Q}W, \mathbf{i} : \mathbb{I}), \qquad \mathbf{Q}(W, \mathbf{i} : \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis) = \mathbf{Q}W.$$

- whose morphisms $\varphi : V \to W$ are morphisms $\mathbf{U}\varphi : \mathbf{U}V \to \mathbf{U}W$ such that if $\mathbf{j}$ has type $\llparenthesis \frown_{\mathrm{eq}} \rrparenthesis$ in $W$, then $\mathbf{j}\langle\varphi\rangle$ only mentions variables $\mathbf{i}$ in $V$ that also have type $\llparenthesis \frown_{\mathrm{eq}} \rrparenthesis$. The morphism $\mathbf{Q}\varphi$ is defined the obvious way.
- where morphisms $\varphi, \chi : V \to W$ are considered equivalent if $\mathbf{Q}\varphi = \mathbf{Q}\chi$.

A setoid-enriched functor $\mathsf{EP} : \mathsf{Cuboid}^{⌑}_M \to \mathsf{Setoid}$ is defined by:

$$\mathsf{EP}() = (), \qquad \mathsf{EP}(W, \mathbf{i} : \mathbb{I}) = \mathsf{EP}(W) \times \{0, 1\}, \qquad \mathsf{EP}(W, \mathbf{i} : \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis) = \mathsf{EP}(W) \times \{0 \frown_{\mathrm{eq}} 1\}.$$

We have a natural transformation $\mathsf{EP} \circ \mathbf{Q} \cong \mathbf{Q} \circ \mathsf{EP}$ of functors $\mathbf{Q}_1 \mathsf{Cuboid}^{⌑}_M \to \mathsf{Set}$.

**Definition 2.5.1°2.** Let $\vec{a}$ be a mask and $\omega$ an orientation kit. We define the set of $(\omega, \vec{a})$**-jet-cuboids** as the set of lists of elements of $\{P_i \mid P \in \omega, i \in \mathsf{Deg}_{\mathrm{eq}}(\vec{a})\}$, where we identify all $P_i = Q_i =: \frown_i$ $(P, Q \in \omega)$ if $a_i = \bigcirc$. We denote jet cuboids as $(\mathbf{i}_0 : \llparenthesis (P_0)_{i_0} \rrparenthesis, \dots, \mathbf{i}_{n-1} : \llparenthesis (P_{n-1})_{i_{n-1}} \rrparenthesis)$, thinking of the names $\mathbf{i}_k$ as De Bruijn indices.

For any $(\omega, \vec{a})$-jet-cuboid $W$, we define $\mathbf{Q}W$ as the $(\omega, \vec{a})$-jet-*cube* obtained by removing all variables of type $\llparenthesis \frown_{\mathrm{eq}} \rrparenthesis$.

**Definition 2.5.1°3.** For any $(\omega, \vec{a})$-jet-cuboid $W$, we define the $\vec{a}$-jet-setoid $\mathsf{JEP}(W)$ as in definition 1.4.2°5 with the additional equation

$$\mathsf{JEP}(W, \mathbf{i} : \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis) = \mathsf{JEP}(W) \times \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis = \mathsf{JEP}(W) * \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis.$$

Clearly, we have $\mathsf{JEP}(\mathbf{Q}W) \cong \mathbf{Q}(\mathsf{JEP}(W))$.

**Definition 2.5.1°4.** For any $(\omega, \vec{a})$-jet-cuboid $W$, we define the jet cuboid $\lfloor W \rfloor$ as in definition 1.4.2°6 with the additional equation

$$\lfloor (W, \mathbf{i} : \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis) \rfloor = (\lfloor W \rfloor, \mathbf{i} : \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis).$$

Clearly, we have $\lfloor \mathbf{Q}W \rfloor = \mathbf{Q}\lfloor W \rfloor$.

**Corollary 2.5.1°5.** For any $(\omega, \vec{a})$-jet-cuboid $W$, we have $U(\mathsf{JEP}(W)) = \mathsf{EP}(\lfloor W \rfloor)$. In other words, corollary 1.4.2°7 remains valid if we replace cubes with cuboids and jet cubes with jet cuboids.

Moreover, the isomorphisms in definition 2.5.1°1 and definition 2.5.1°3 are compatible along $U : \mathsf{JetSet}(\vec{a}) \to \mathsf{Set}$ and $\lfloor \sqcup \rfloor$. $\square$

**Definition 2.5.1°6.** Let $\vec{a}$ be a mask, $\omega \in \mathsf{OKit}$, $⌑ \in \{\square, \boxslash\}$ and $M$ a monad on $\mathsf{Set}$. We define the setoid-enriched category $\mathsf{JetCuboid}^{⌑}_M(\omega, \vec{a})$ of $(\omega, \vec{a})$**-jet-$M$-cuboids** as the full subcategory of $\mathsf{Cuboid}^{⌑}_M \times_{\mathsf{Set}} \mathsf{JetSet}(\vec{a})$ on $(\omega, \vec{a})$-jet-cuboids, as justified by corollary 2.5.1°5. The functions $\mathsf{JEP}$ and $\lfloor \sqcup \rfloor$ are correspondingly extended to setoid-enriched functors.

**Corollary 2.5.1°7.** In the following cube-shaped diagram, the side squares commute strictly while the top and bottom squares commute up to compatible natural isomorphisms:

$$\begin{tikzcd}
\mathsf{JetCuboid}^{⌑}_M(\omega, \vec{a}) \arrow[rrr, "\mathsf{JEP}"] \arrow[dr, "\mathbf{Q}"] \arrow[dd, "\lfloor \sqcup \rfloor"'] & & & \mathsf{JetSetoid}(\vec{a}) \arrow[dr, "\mathbf{Q}"] \arrow[dd, "U"'] & \\
& \mathsf{JetCube}^{⌑}_M(\omega, \vec{a}) \arrow[rrr, "\mathsf{JEP}"] \arrow[dd, "\lfloor \sqcup \rfloor"'] & & & \mathsf{JetSet}(\vec{a}) \arrow[dd, "U"] \\
\mathsf{Cuboid}^{⌑}_M \arrow[rrr, "\mathsf{EP}"] \arrow[dr, "\mathbf{Q}"] & & & \mathsf{Setoid} \arrow[dr, "\mathbf{Q}"] & \\
& \mathsf{Cube}^{⌑}_M \arrow[rrr, "\mathsf{EP}"] & & & \mathsf{Set}.
\end{tikzcd}$$

By the adjunction $\mathbf{Q}_1 \dashv \mathbf{D}_1$ (definition 2.2.0°3), we can regard each of the arrows labelled $\mathbf{Q}$ in the diagram above as either starting from $\mathbf{Q}_1$ of the domain (making it an ordinary functor) or arriving at $\mathbf{D}_1$ of the codomain (making it a setoid-enriched functor). The arrows in the upper left square parallel to the page are all setoid-enriched functors, while the arrows in the lower right square are all ordinary functors. $\square$

**Definition 2.5.1°8.** We define a setoid-enriched functor $\mathbf{D} : \mathbf{D}_1\mathsf{Cube}^{\#}_M \to \mathsf{Cuboid}^{\#}_M$ which sends a cube $W = (\mathbf{i}_0 : \mathbb{I}, \ldots, \mathbf{i}_{n-1} : \mathbb{I})$ to the cuboid $\mathbf{D}W = (\mathbf{i}_0 : \mathbb{I}, \ldots, \mathbf{i}_{n-1} : \mathbb{I})$ of the same notation, and has an obvious action on morphisms.

**Corollary 2.5.1°9.** $\mathbf{D} : \mathsf{Cube}^{\#}_M \to \mathbf{U}_1\mathsf{Cuboid}^{\#}_M$ is right adjoint to $\mathbf{Q} : \mathbf{U}_1\mathsf{Cuboid}^{\#}_M \to \mathsf{Cube}^{\#}_M$. $\square$

**Corollary 2.5.1°10.** $\mathbf{D} : \mathbf{D}_1\mathsf{Cube}^{\#}_M \to \mathsf{Cuboid}^{\#}_M$ and $\mathbf{Q} : \mathsf{Cuboid}^{\#}_M \to \mathbf{D}_1\mathsf{Cube}^{\#}_M$ form an equivalence of setoid-enriched categories, where $\mathbf{DQ}$ is naturally *equivalent* but not naturally isomorphic to the identity. $\square$

**Definition 2.5.1°11.** We define a setoid-enriched functor $\mathbf{D} : \mathbf{D}_1\mathsf{JetCube}^{\#}_M(\omega, \vec{a}) \to \mathsf{JetCuboid}^{\#}_M(\omega, \vec{a})$ which sends a jet cube $W = (\mathbf{i}_0 : (\!|(P_0)_{i_0}|\!), \ldots, \mathbf{i}_{n-1} : (\!|(P_{n-1})_{i_{n-1}}|\!))$ to the jet cuboid $\mathbf{D}W = (\mathbf{i}_0 : (\!|(P_0)_{i_0}|\!), \ldots, \mathbf{i}_{n-1} : (\!|(P_{n-1})_{i_{n-1}}|\!))$ of the same notation, and has an obvious action on morphisms.

**Corollary 2.5.1°12.** $\mathbf{D} : \mathsf{JetCube}^{\#}_M(\omega, \vec{a}) \to \mathbf{U}_1\mathsf{JetCuboid}^{\#}_M(\omega, \vec{a})$ is right adjoint to $\mathbf{Q} : \mathbf{U}_1\mathsf{JetCuboid}^{\#}_M(\omega, \vec{a}) \to \mathsf{JetCube}^{\#}_M(\omega, \vec{a})$. $\square$

**Corollary 2.5.1°13.** $\mathbf{D} : \mathbf{D}_1\mathsf{JetCube}^{\#}_M(\omega, \vec{a}) \to \mathsf{JetCuboid}^{\#}_M(\omega, \vec{a})$ and $\mathbf{Q} : \mathsf{JetCuboid}^{\#}_M(\omega, \vec{a}) \to \mathbf{D}_1\mathsf{JetCube}^{\#}_M(\omega, \vec{a})$ form an equivalence of setoid-enriched categories, where $\mathbf{DQ}$ is naturally *equivalent* but not naturally isomorphic to the identity. $\square$

**Corollary 2.5.1°14.** There is a strictly commutative diagram

$$\begin{tikzcd}
\mathsf{JetCuboid}^{\#}_M(\omega,\vec{a}) \arrow[rr, "\mathsf{JEP}"] \arrow[dd, "\lfloor\sqcup\rfloor"'] & & \mathsf{JetSetoid}(\vec{a}) \arrow[dd, "U"] & \\
& \mathsf{JetCube}^{\#}_M(\omega,\vec{a}) \arrow[ul, "\mathbf{D}"] \arrow[rr, "\mathsf{JEP}"] \arrow[dd, "\lfloor\sqcup\rfloor"'] & & \mathsf{JetSet}(\vec{a}) \arrow[ul, "\mathbf{D}"'] \arrow[dd, "U"] \\
\mathsf{Cuboid}^{\#}_M \arrow[rr, "\mathsf{EP}"] & & \mathsf{Setoid} & \\
& \mathsf{Cube}^{\#}_M \arrow[ul, "\mathbf{D}"] \arrow[rr, "\mathsf{EP}"] & & \mathsf{Set}. \arrow[ul, "\mathbf{D}"']
\end{tikzcd}$$

where the arrows in the upper left square parallel to the page are setoid-enriched functors, the arrows in the lower right square are ordinary functors, and the arrows labelled $\mathbf{D}$ can be regarded as setoid-enriched functors from $\mathbf{D}_1$ of the domain or, equivalently by definition 2.2.0°3, as ordinary functors to $\mathbf{U}_1$ of the codomain. $\square$

#### 2.5.1 (b) Semantics on Jet Cubes and Jet Cuboids

**Definition 2.5.1°15.** 1. We define a strict 2-functor $\mathsf{JetCuboid}^{\square}_M(\mathsf{fb}, \sqcup) : \mathsf{PoResh}^{\bullet\circ\circ} \to \mathsf{SetoidCat}$ which sends:

- an object $\vec{a}$ to $\mathsf{JetCuboid}^{\square}_M(\mathsf{fb}, \vec{a})$,
- a morphism $\mu : \vec{a} \to \vec{b}$ with right adjoint $\mu \dashv \rho$ to a functor $\mathsf{JetCuboid}^{\square}_M(\mathsf{fb}, \mu)$ (also just denoted $\mu$) which:
  - sends $()$ to $\mu() := ()$,

  – has an action on other cuboids that can be looked up in the following table:[13,14]

| $i \cdot \rho$ | $\odot_{\mathrm{eq}}$ | $\oplus_j$ | $\ominus_j$ |
|---|---|---|---|
| $\mu(W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$ | $(\mu W, \mathbf{i} : \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis)$ | $(\mu W, \mathbf{i} : \llparenthesis \rightarrowtriangle_j \rrparenthesis)$ | $(\mathsf{Op}_j \mu W, \mathbf{i} : \llparenthesis \leftarrowtriangle_j \rrparenthesis)$ |
| $\mu(W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis)$ | $(\mu W, \mathbf{i} : \llparenthesis \frown_{\mathrm{eq}} \rrparenthesis)$ | $(\mu W, \mathbf{i} : \llparenthesis \leftarrowtriangle_j \rrparenthesis)$ | $(\mathsf{Op}_j \mu X, \mathbf{i} : \llparenthesis \rightarrowtriangle_j \rrparenthesis)$ |

  – thus satisfies $\mathsf{JEP}(\mu W) = \mu\mathsf{JEP}(W)$ by proposition 2.2.1°6,
  – has an action on jet cuboid morphisms defined as follows. Recall that a jet cuboid morphism $\varphi : V \to W$ is a cuboid morphism $\lfloor \varphi \rfloor : \lfloor V \rfloor \to \lfloor W \rfloor$ whose endpoint model $\mathsf{EP}(\lfloor \varphi \rfloor) = U(\mathsf{JEP}(\varphi))$ is in fact a morphism of jet sets $\mathsf{JEP}(\varphi) : \mathsf{JEP}(V) \to \mathsf{JEP}(W)$ (definition 2.5.1°6).
    * We define $\mathbf{j}\langle \lfloor \mu(\varphi) \rfloor \rangle := \mathbf{j}\langle \lfloor \varphi \rfloor \rangle$.
    * It is the case that

$$\mathsf{EP}(\lfloor \mu(\varphi) \rfloor) : \mathsf{EP}(\lfloor \mu V \rfloor) = U(\mathsf{JEP}(\mu V)) \to U(\mathsf{JEP}(\mu W)) = \mathsf{EP}(\lfloor \mu W \rfloor)$$

    is in fact a morphism of jet setoids; indeed it is the morphism $\mu(\mathsf{JEP}(\varphi))$, as can be trivially proven by induction on $W$ of $\varphi$.
  – If $\mu \leq \nu$, then we have to give a morphism $[\mu \leq \nu\rangle : \mu W \to \nu W$, naturally w.r.t. $W$. We define $\lfloor [\mu \leq \nu\rangle \rfloor$ ny $\mathbf{i}\langle \lfloor [\mu \leq \nu\rangle \rfloor \rangle = \mathbf{i}$, which is a valid morphism of cuboids exactly because we know that $\mu \leq \nu$. This is clearly natural on cubes, therefore on sets, and therefore on jet setoids. Its endpoint model (EP) is a jet setoid morphism, again because $\mu \leq \nu$.

2. By corollaries 2.2.0°4, 2.2.0°7, 2.5.1°10 and 2.5.1°13, we obtain a pseudofunctor $\mathsf{JetCube}^{\Box}_M(\mathsf{fb}, \sqcup) : \mathsf{PoResh}^{\bullet\circ\circ} \to \mathsf{Cat}$:
   - Sending an object $\vec{a}$ to $\mathsf{JetCube}^{\Box}_M(\mathsf{fb}, \vec{a})$,
   - Sending a morphism $\mu : \vec{a} \to \vec{b}$ to $\mathsf{JetCube}^{\Box}_M(\mathsf{fb}, \mu) = \mathbf{Q} \circ \mathsf{JetCuboid}^{\Box}_M(\mathsf{fb}, \mu) \circ \mathbf{D} : \mathsf{JetCube}^{\Box}_M(\mathsf{fb}, \vec{a}) \to \mathsf{JetCube}^{\Box}_M(\mathsf{fb}, \vec{b})$,
   - …
3. Additionally, if we consider only modalities $\mu$ satisfying eq. (2.6) (i.e. those for which $\mathrm{eq} \cdot \mu = \odot_{\mathrm{eq}}$) and denote the restricted 2-poset as $\mathsf{PoResh}^{[\mathrm{eq}]\bullet\circ\circ}$, then we directly get a strict 2-functor $\mathsf{JetCube}^{\Box}_M(\mathsf{fb}, \sqcup)^{[\mathrm{eq}]} : \mathsf{PoResh}^{[\mathrm{eq}]\bullet\circ\circ} \to \mathsf{Cat}$, equivalent to the pseudofunctor above:
   - Sending an object $\vec{a}$ to $\mathsf{JetCube}^{\Box}_M(\mathsf{fb}, \vec{a})$,
   - Sending a morphism $\mu : \vec{a} \to \vec{b}$ to $\mathsf{JetCube}^{\Box}_M(\mathsf{fb}, \mu)^{[\mathrm{eq}]}$ which acts on cubes the same way $\mathsf{JetCuboid}^{\Box}_M(\mathsf{fb}, \mu)$ acts on cuboids.
   - …
4. By proposition 1.4.2°10, if $M \in \{\mathsf{IPt}_2, \mathsf{Boo}\}$, we obtain pseudofunctors

$$\mathsf{JetCube}^{\Box}_M(\mathsf{f}, \sqcup) : \mathsf{PoResh}^{\bullet\circ\circ} \to \mathsf{Cat},$$
$$\mathsf{JetCuboid}^{\Box}_M(\mathsf{f}, \sqcup) : \mathsf{PoResh}^{\bullet\circ\circ} \to \mathsf{SetoidCat}.$$

### 2.5.2 Adjoint Triple Semantics on Jet Cubical Sets

**Definition 2.5.2°1.** The category of $(\omega, \vec{a})$-**jet-cubical sets** is the presheaf category $\mathsf{JCSet}^{\Box}_M(\omega, \vec{a}) := \mathsf{Psh}(\mathsf{JetCube}^{\Box}_M(\omega, \vec{a}))$.

**Theorem 2.5.2°2.** Let $\omega \in \{\mathsf{f}, \mathsf{fb}\}$ and, if $\omega = \mathsf{f}$ then $M \in \{\mathsf{IPt}_2, \mathsf{Boo}\}$. We have the following pseudofunctors:

- $\mathsf{JCSet}^{\Box}_M(\omega, \sqcup)^{\bullet\circ\circ} : \mathsf{PoResh}^{\bullet\circ\circ} \to \mathsf{Cat}^{\bullet\circ\circ}$,

[13] Note that if $b_j = \bigcirc$, then the results for $\odot_j$ are consistent.
[14] Note that if $i \cdot \rho \in \{\otimes_j, \circledast_j\}$, then by eqs. (2.7) and (2.9), $b_j = \bigcirc$.

- $\mathsf{JCSet}^{\square}_{M}(\omega, \sqcup)^{\circ\bullet\circ} : \mathsf{PoResh}^{\circ\bullet\circ} \to \mathsf{Cat}^{\circ\bullet\circ}$,
- $\mathsf{JCSet}^{\square}_{M}(\omega, \sqcup)^{\circ\circ\bullet} : \mathsf{PoResh}^{\circ\circ\bullet} \to \mathsf{Cat}^{\circ\circ\bullet}$,

to the 2-category of categories, functors with a specified number of left/right adjoints (notation 2.1.3°1), and natural transformations, where:

- each functor sends an object $\vec{a}$ to $\mathsf{Psh}(\mathsf{JetCube}^{\square}_{M}(\omega, \vec{a}))$,
- a triple of polarized reshuffles $\mu \dashv \rho \dashv \tau$ is sent by the respective functors to

$$\mathsf{JCSet}^{\square}_{M}(\omega, \mu)^{\bullet\circ\circ} = \mu_! \quad \dashv \quad \mathsf{JCSet}^{\square}_{M}(\omega, \rho)^{\circ\bullet\circ} = \mu^* \quad \dashv \quad \mathsf{JCSet}^{\square}_{M}(\omega, \tau)^{\circ\circ\bullet} = \mu_*,$$

  of which $\mu^*$ is a strict CwF morphism [Dyb95] and $\mu_*$ is a weak CwF morphism [Nuy18, def. 2.1.1][Nuy20a, def. 3.2.5] w.r.t. the standard CwF structure on a presheaf category [Hof97, ch. 4][Nuy20a, §4.1].
- 2-cells are also respected.

*Proof.* By the exact same reasoning as in the proof of theorem 2.4.2°2. □

### 2.5.3 No Adjoint Pair Semantics on Jet Cubical Sets

As in section 2.4.3, we have a no-go theorem:

**Theorem 2.5.3°1.** Let $\omega \in \{\mathsf{f}, \mathsf{fb}\}$ and $M \in \{\mathsf{IPt}_2, \mathsf{Boo}\}$. The following, equivalent[15], objects **do not exist**:

- A pseudofunctor $\mathsf{JCSet}^{\square}_{M}(\omega, \sqcup)^{\bullet\circ} : \mathsf{PoResh}^{\bullet\circ} \to \mathsf{Cat}^{\bullet\circ}$ that extends both $\mathsf{JCSet}^{\square}_{M}(\omega, \sqcup)^{\bullet\circ\circ}$ and $\mathsf{JCSet}^{\square}_{M}(\omega, \sqcup)^{\circ\bullet\circ}$ up to natural isomorphism,
- A pseudofunctor $\mathsf{JCSet}^{\square}_{M}(\omega, \sqcup)^{\circ\bullet} : \mathsf{PoResh}^{\circ\bullet} \to \mathsf{Cat}^{\circ\bullet}$ that extends both $\mathsf{JCSet}^{\square}_{M}(\omega, \sqcup)^{\circ\bullet\circ}$ and $\mathsf{JCSet}^{\square}_{M}(\omega, \sqcup)^{\circ\circ\bullet}$ up to natural isomorphism.

*Proof.* The reasoning in the proof of theorem 2.4.3°1, still stands for jet cubical sets. Note:

- That the interval jet set $(\!|{\multimapdotinv}_0|\!)$ needs to be replaced with the 1-dimensional jet cube $(\mathbf{i} : (\!|{\multimapdotinv}_0|\!))$.
- That all jet set morphisms of JEPs of jet cubes, whose domain is 1-dimensional, are in fact jet cube morphisms. □

### 2.5.4 Conclusion

We draw the same conclusion as in section 2.4.4.

[15] Assuming an adjoint can be picked if one exists.

# Chapter 3

# Paths and Bridges

$\langle \odot_{\mathrm{eq}} \,|\, \otimes_0 \rangle : [\nearrow] \to [\nearrow]$ **is not part of a chain of 3 and therefore unincludable. This is a strong argument for bridges and paths.** □

## 3.1 Rods, Paths and Bridges

**Reabolish rods** □

In this section, we define a number of operations on anpolarity masks (definition 1.1.0°1) which serve to insert symmetric *bridge* and/or *path* relations for every degree. The idea, in the universe of small types, is the following:

- Rods express strict equality,
- Paths express isomorphism (in the presence of rods or bridges) or equivalence (in absence of rods),
- Jets express homomorphism,
- Bridges express relations.

Bridges and rods will not occur together, because bridges can be regarded as pro-rods (with 'pro-' in the sense of pro-arrow equipments): a bridge between types expresses a relation, which is a notion of heterogeneous strict equality.

The mask operations serve three goals:

- They are helpful to understand equijets[1],
- They generate the modes that we will likely want to use in practice.

**Definition 3.1.0°1.** Let $\mathsf{X} \in \{\mathsf{J}, \mathsf{PJ}, \mathsf{JB}, \mathsf{PJB}, \mathsf{RPJ}\}$ and $\mathsf{Y} \in \{\mathsf{E}, \mathsf{PB}, \mathsf{RP}\}$, and let $a$ be an anpolarity. We define $a^{\mathsf{X};\mathsf{Y}} \in \mathsf{List}\,\mathbb{A}$ as follows:

$$\begin{array}{rclcrcl}
\nearrow^{\mathsf{J};\mathsf{Y}} & = & [\nearrow] & \qquad & \bigcirc^{\mathsf{X};\mathsf{E}} & = & [\bigcirc] \\
\nearrow^{\mathsf{PJ};\mathsf{Y}} & = & [\bigcirc, \nearrow] & & \bigcirc^{\mathsf{X};\mathsf{PB}} & = & [\bigcirc, \bigcirc] \\
\nearrow^{\mathsf{JB};\mathsf{Y}} & = & [\nearrow, \bigcirc] & & \bigcirc^{\mathsf{X};\mathsf{RP}} & = & [\bigcirc, \bigcirc] \\
\nearrow^{\mathsf{PJB};\mathsf{Y}} & = & [\bigcirc, \nearrow, \bigcirc] & & & & \\
\nearrow^{\mathsf{RPJ};\mathsf{Y}} & = & [\bigcirc, \bigcirc, \nearrow] & & & &
\end{array}$$

We extend this operation to anpolarity masks as follows: $\vec{a}^{\mathsf{X};\mathsf{Y}}$ is the concatenation (join / monadic multiplication) of the list of lists obtained by applying $\sqcup^{\mathsf{X};\mathsf{Y}}$ to every element of $\vec{a}$.

Indexing the resulting mask is complicated, because the length of the resulting mask depends not only on the length of $\vec{a}$ but also on its contents. In order to stay sane, we will use an indexing system that still refers to the degrees of $\vec{a}$, which we will call **big degrees**, whereas the degrees of $\vec{a}^{\mathsf{X};\mathsf{Y}}$ will be called **small degrees**. Concretely, a small degree subscript will be replaced with a big degree subscript

[1] refer

combined with a superscript indicating whether we are talking about the rod/path/jet/bridge or edge relation. For example, if $\vec{a} = [\times\!\!\!\!\nearrow, \bigcirc, \times\!\!\!\!\nearrow]$, then the relations of $\vec{a}^{\mathsf{PJB;E}}$ will be denoted as:

$$\frown_0^{\mathsf{P}}, \quad \rightarrowtriangle_0^{\mathsf{J}}, \quad \frown_0^{\mathsf{B}}, \quad \frown_1^{\mathsf{E}}, \quad \frown_2^{\mathsf{P}}, \quad \rightarrowtriangle_2^{\mathsf{J}}, \quad \frown_2^{\mathsf{B}},$$

rather than

$$\frown_0, \quad \rightarrowtriangle_1, \quad \frown_2, \quad \frown_3, \quad \frown_4, \quad \rightarrowtriangle_5, \quad \frown_6.$$

When we need a degree in itself, we will write e.g. $0^{\mathsf{P}}$.

A modality $\mu : \vec{a}^{\mathsf{PJB;E}} \to \vec{a}^{\mathsf{PJB;E}}$ will be denoted

$$\left\langle \mathrm{eq} \cdot \mu \,\middle|\, 0^{\mathsf{P}} \cdot \mu, 0^{\mathsf{J}} \cdot \mu, 0^{\mathsf{B}} \cdot \mu; 1^{\mathsf{E}} \cdot \mu; 2^{\mathsf{P}} \cdot \mu, 2^{\mathsf{J}} \cdot \mu, 2^{\mathsf{B}} \cdot \mu \right\rangle,$$

i.e. we will use semicolons (;) to group small degree information per big degree.

### 3.1.1 Understanding Equijets

**This is way easier if you have modalities** □

# Chapter 4

# Transpension

**Does $\mathrm{Tw} : \mathrm{sSet} \to \mathrm{sSet}$ have a transp type? (Yes if it's $\mathrm{Tw}_!$)** $\square$

# Appendix A

# Version history

**v0.1** Wrote chapter 1.
**v0.1.1** Changed title.
Proofread chapter 1.
Added missing clause for CONCURSOR in proof of theorem 1.4.3°4.
Cleaned up affine boolean normalization lemmas in section 1.4.3 (b).
Introduced orientation kits (definition 1.4.2°1).
Introduced conventional cubes (definition 1.4.3°1).
General cleanup.
**v0.2** Wrote chapter 2.
Wrote section 3.1.
Changed title.
Renamed jet jewels to jet topes; modified their definition (definition 1.3.0°2).

# Bibliography


[AAG05] Michael Abbott, Thorsten Altenkirch, and Neil Ghani. Containers: Constructing strictly positive types. *Theoretical Computer Science*, 342(1):3–27, 2005. Applied Semantics: Selected Topics. URL: https://www.sciencedirect.com/science/article/pii/S0304397505003373, doi:https://doi.org/10.1016/j.tcs.2005.06.002.

[Abe08] Andreas Abel. Polarised subtyping for sized types. *Mathematical Structures in Computer Science*, 18(5):797–822, 2008. doi:10.1017/S0960129508006853.

[ARVL10] J. Adámek, J. Rosický, E. M. Vitale, and F. W. Lawvere. *Algebraic Theories: A Categorical Introduction to General Algebra*. Cambridge Tracts in Mathematics. Cambridge University Press, 2010. doi:10.1017/CBO9780511760754.

[BCH14] Marc Bezem, Thierry Coquand, and Simon Huber. A Model of Type Theory in Cubical Sets. In *19th International Conference on Types for Proofs and Programs (TYPES 2013)*, volume 26, pages 107–128, Dagstuhl, Germany, 2014. URL: http://drops.dagstuhl.de/opus/volltexte/2014/4628, doi:10.4230/LIPIcs.TYPES.2013.107.

[BCM15] Jean-Philippe Bernardy, Thierry Coquand, and Guilhem Moulin. A presheaf model of parametric type theory. *Electron. Notes in Theor. Comput. Sci.*, 319:67 – 82, 2015. doi:http://dx.doi.org/10.1016/j.entcs.2015.12.006.

[BCM+20] Lars Birkedal, Ranald Clouston, Bassel Mannaa, Rasmus Ejlers Møgelberg, Andrew M. Pitts, and Bas Spitters. Modal dependent type theory and dependent right adjoints. *Math. Struct. Comput. Sci.*, 30(2):118–138, 2020. doi:10.1017/S0960129519000197.

[CCHM15] Cyril Cohen, Thierry Coquand, Simon Huber, and Anders Mörtberg. Cubical type theory: A constructive interpretation of the univalence axiom. In Tarmo Uustalu, editor, *21st International Conference on Types for Proofs and Programs, TYPES 2015, May 18-21, 2015, Tallinn, Estonia*, volume 69 of *LIPIcs*, pages 5:1–5:34. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2015. doi:10.4230/LIPIcs.TYPES.2015.5.

[Dyb95] Peter Dybjer. Internal type theory. In Stefano Berardi and Mario Coppo, editors, *Types for Proofs and Programs, International Workshop TYPES'95, Torino, Italy, June 5-8, 1995, Selected Papers*, volume 1158 of *Lecture Notes in Computer Science*, pages 120–134. Springer, 1995. doi:10.1007/3-540-61780-9“66.

[GKNB21] Daniel Gratzer, G. A. Kavvos, Andreas Nuyts, and Lars Birkedal. Multimodal Dependent Type Theory. *Logical Methods in Computer Science*, Volume 17, Issue 3, July 2021. URL: https://lmcs.episciences.org/7713, doi:10.46298/lmcs-17(3:11)2021.

[Hof97] Martin Hofmann. *Syntax and Semantics of Dependent Types*, pages 79–130. Cambridge University Press, 1997.

[Man12] Ernest G Manes. *Algebraic theories*, volume 26 of *Graduate Texts in Mathematics*. Springer Science & Business Media, 2012.

[ND18] Andreas Nuyts and Dominique Devriese. Degrees of relatedness: A unified framework for parametricity, irrelevance, ad hoc polymorphism, intersections, unions and algebra in dependent type theory. In *Logic in Computer Science (LICS) 2018, Oxford, UK, July 09-12, 2018*, pages 779–788, 2018. URL: `https://lirias.kuleuven.be/retrieve/510956`, `doi:10.1145/3209108.3209119`.

[nLaa] nLab authors. 2-category equipped with proarrows, April. Revision 32. URL: `http://ncatlab.org/nlab/show/2-category%20equipped%20with%20proarrows`.

[nLab] nLab authors. adjoint functor, June. URL: `https://ncatlab.org/nlab/show/adjoint+functor`.

[nLac] nLab authors. functoriality of categories of presheaves, February. Revision 6. URL: `https://ncatlab.org/nlab/show/functoriality+of+categories+of+presheaves`.

[Nuy18] Andreas Nuyts. Presheaf models of relational modalities in dependent type theory. *CoRR*, abs/1805.08684, 2018. `arXiv:1805.08684`.

[Nuy20a] Andreas Nuyts. *Contributions to Multimode and Presheaf Type Theory*. PhD thesis, KU Leuven, Belgium, 8 2020. URL: `https://lirias.kuleuven.be/3065223`.

[Nuy20b] Andreas Nuyts. The transpension type: Technical report, 2020. URL: `https://arxiv.org/abs/2008.08530`, `arXiv:2008.08530`.

[Nuy22] Andreas Nuyts. Understanding universal algebra using Kleisli-Eilenberg-Moore-Lawvere diagrams. Unpublished note, 2022. URL: `https://anuyts.github.io/files/keml-diagrams.pdf`.

[PEC+23] Josselin Poiret, Lucas Escot, Joris Ceulemans, Malin Altenmüller, and Andreas Nuyts. Read the mode and stay positive. In *29th International Conference on Types for Proofs and Programs (TYPES)*, 6 2023. URL: `https://lirias.kuleuven.be/retrieve/720869`.

[PK20] Gun Pinyo and Nicolai Kraus. From Cubes to Twisted Cubes via Graph Morphisms in Type Theory. In Marc Bezem and Assia Mahboubi, editors, *25th International Conference on Types for Proofs and Programs (TYPES 2019)*, volume 175 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 5:1–5:18, Dagstuhl, Germany, 2020. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. URL: `https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.TYPES.2019.5`, `doi:10.4230/LIPIcs.TYPES.2019.5`.

[Shu23] Michael Shulman. Semantics of multimodal adjoint type theory. In Marie Kerjean and Paul Blain Levy, editors, *Proceedings of the 39th Conference on the Mathematical Foundations of Programming Semantics, MFPS XXXIX, Indiana University, Bloomington, IN, USA, June 21-23, 2023*, volume 3 of *EPTICS*. EpiSciences, 2023. URL: `https://doi.org/10.46298/entics.12300`, `doi:10.46298/ENTICS.12300`.

[Sta] The Stacks Project Authors. Stacks project. `http://stacks.math.columbia.edu`. Tags 00VC and 00XF.

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `http://homotopytypetheory.org/book`, IAS, 2013.

[Uus17] Tarmo Uustalu. Container combinatorics: Monads and lax monoidal functors. In Mohammad Reza Mousavi and Jiří Sgall, editors, *Topics in Theoretical Computer Science*, pages 91–105, Cham, 2017. Springer International Publishing.

[Woo82] Richard J Wood. Abstract pro arrows I. *Cahiers de topologie et géométrie différentielle*, 23(3):279–290, 1982.

[Woo85] Richard J Wood. Proarrows II. *Cahiers de topologie et géométrie différentielle catégoriques*, 26(2):135–168, 1985.