Naturality Pretype Theory: Technical Report v0.1.1

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Chapter 1

Base Categories and Modes

In this chapter, we establish the object part of the mode theory for naturality pretype theory and its model. Modes will be **anpolarity masks** $\vec{a}$ (section 1.1) which will be modelled in presheaf categories over certain base categories. We propose two families of base categories, parametrized by the mask $\vec{a}$, and both are built using the category $\mathsf{JetSet}(\vec{a})$ of **jet sets** over $\vec{a}$ (section 1.2).

The more complex but better behaved family of base categories are categories of **jet cubes** $\mathsf{JetCube}^{\square}_{M}(\mathsf{f}, \vec{a})$ (section 1.4). While we can get a grasp on jet cubes by characterizing their morphisms using a calculus (section 1.4.3), this calculus is relatively complex and its soundness and completeness proofs are even more complex.

For these reasons, we alternatively propose to use categories of **jet jewels** $\mathsf{JetJewel}(\vec{a})$ (section 1.3). These are full subcategories of $\mathsf{JetSet}(\vec{a})$ on objects that satisfy certain somewhat arbitrary well-behavedness criteria. The main purpose of the categories of jet jewels is to be workable for most basic purposes without being as complex as the categories of jet cubes.

Upon a first lecture, readers may choose to omit section 1.4 on jet cubes.

I momentarily had some doubt about usability of jet jewels, but I think it's okay. The calculus and its soundness proof are used mainly for modalities interacting with the interval. But these will not be used intensively in the NatPT paper itself. $\square$

1.1 Anpolarity Masks

In RelDTT, modes were natural numbers (minus one) expressing the number of available relations. In NatPT, we will specify for each of these relations whether it is directed or not.

Definition 1.1.0°1. An **anpolarity**[1] is an element of the set $\mathbb{A} := \{\mathrlap{\times}\nearrow, \bigcirc\}$, where $\mathrlap{\times}\nearrow$ stands for polar/directed and $\bigcirc$ stands for nonpolar/symmetric. We equip $\mathbb{A}$ with the partial order $\bigcirc \sqsubseteq \mathrlap{\times}\nearrow$, corresponding to the intuition that symmetric relations are a subset of directed (i.e. potentially asymmetric) relations.

An **anpolarity mask** or just **mask** is a list $\vec{a} \in \mathsf{List}\,\mathbb{A}$ of anpolarities. We write $\mathsf{len}(\vec{a})$ for its length, and call the numbers $0, \ldots, \mathsf{len}(\vec{a}) - 1$ **degrees**. We define $\sqsubseteq$ on masks of equal length pointwise.

Both for anpolarities and for masks of equal length, we denote meets (infima) with $\sqcap$ and joins (suprema) with $\sqcup$.

1.2 Jet Sets

1.2.1 Definitions

Definition 1.2.1°1. Let $\vec{a}$ be a mask. An **$\vec{a}$-jet-set** is a set X equipped with $\mathsf{len}(\vec{a})$ (proof-irrelevant[2]) reflexive relations $\rightarrowtriangle_i$ where

[1] 'An' is Latin for 'whether', as in 'Nescio an polare sit,' meaning 'I do not know whether it is polar'.

[2] So these relations are functions $X \to X \to \mathsf{Prop}$ where Prop is a universe of h-propositions [Uni13]. In most applications, these relations will be decidable, but we do not require this.

- $0 \leq i < \mathsf{len}(\vec{a})$ is called the **degree**,
- $\rightarrowtriangle_i$ is called the i**-jet relation**,
- its opposite $\leftarrowtriangle_i$ is called the **opposite** i**-jet relation**,

such that

- when $a_i = \bigcirc$, then $\rightarrowtriangle_i$ is symmetric, in which case we will denote it as $\frown_i$ and call it the i**-edge relation** (notwithstanding that we still consider it a special case of a jet relation),
- $x \rightarrowtriangle_i y$ implies both $x \rightarrowtriangle_{i+1} y$ and $x \leftarrowtriangle_{i+1} y$ whenever $0 \leq i < i+1 < \mathsf{len}(\vec{a})$.

A **morphism** of $\vec{a}$-jet-sets is a function that preserves all the jet and edge relations.

The category of $\vec{a}$-jet-sets is called $\mathsf{JetSet}(\vec{a})$.

Definition 1.2.1°2. A jet set morphism is called **full** if it reflects all jet relations.

Definition 1.2.1°3. A jet set morphism $f : X \to Y$ is called **jet-surjective** if it is surjective as a function, and moreover for any $\vec{y} \rightarrowtriangle_j \vec{y}'$ in Y, there exist $\vec{x} \rightarrowtriangle_j \vec{x}'$ in X such that $f(\vec{x}) = \vec{y}$ and $f(\vec{x}') = \vec{y}'$.

Definition 1.2.1°4. A jet set is called **transitive** if each of the i-jet relations is transitive (i.e. a pre-order and, if $i = \bigcirc$, an equivalence relation).

The following proposition will not really be used directly, but is a nice encouragement:

Proposition 1.2.1°5. Let X be a transitive $\vec{a}$-jet-set and $0 \leq i < j < \mathsf{len}(\vec{a})$. Then the double category whose objects are elements of X, morphisms are (unique) proofs of $x \rightarrowtriangle_i y$, pro-arrows are (unique) proofs of $x \rightarrowtriangle_j y$ and squares are elements of the unit type, is a pro-arrow equipment [nLa20, Woo82, Woo85].

Proof. It is clearly a double category. The existence of companions and conjoints is trivial. □

Definition 1.2.1°6. We define the

- i-**equijet relation** $\leftrightarrowtriangle_i$ as the symmetric interior of $\rightarrowtriangle_i$, i.e. $x \leftrightarrowtriangle_i y$ if and only if $x \rightarrowtriangle_i y$ and $x \leftarrowtriangle_i y$;
- i-**infrajet relation** $\rightleftarrows_i$ as the symmetric closure of $\rightarrowtriangle_i$, i.e. $x \rightleftarrows_i y$ if and only if $x \rightarrowtriangle_i y$ or $x \leftarrowtriangle_i y$.

It is immediately clear that for nonpolar degrees, the jet/edge, equijet and infrajet relations coincide. In general, we can observe that $x \rightleftarrows_i y$ implies $x \leftrightarrowtriangle_j y$ for $i < j$. So for mode $[\nearrow, \nearrow, \bigcirc]$, we get

$$\begin{tikzcd}
 & \rightarrowtriangle_0 \arrow[dr, dotted] & & & \rightarrowtriangle_1 \arrow[dr, dotted] & & & \rightarrowtriangle_2 \arrow[dl, leftrightarrow, dotted] \arrow[d, leftrightarrow, dotted] \arrow[dr, leftrightarrow, dotted] & \\
\leftrightarrowtriangle_0 \arrow[ur, dotted] \arrow[dr, dotted] & & \rightleftarrows_0 \arrow[r, dotted] & \leftrightarrowtriangle_1 \arrow[ur, dotted] \arrow[dr, dotted] & & \rightleftarrows_1 \arrow[r, dotted] & \leftrightarrowtriangle_2 \arrow[r, leftrightarrow, dotted] & \frown_2 \arrow[r, leftrightarrow, dotted] & \rightleftarrows_2 \\
 & \leftarrowtriangle_0 \arrow[ur, dotted] & & & \leftarrowtriangle_1 \arrow[ur, dotted] & & & \leftarrowtriangle_2 \arrow[ul, leftrightarrow, dotted] \arrow[u, leftrightarrow, dotted] \arrow[ur, leftrightarrow, dotted] &
\end{tikzcd}.$$

Definition 1.2.1°7. Let $\vec{a}$ be a mask, $i < \mathsf{len}(\vec{a})$ and $X \in \mathrm{Obj}(\mathsf{JetSet}(\vec{a}))$. We define the i**-opposite** $\mathsf{Op}_i(X)$ **of** X as the jet set with the same carrier and relations as X except that the i-jet relation is reversed: $x \rightarrowtriangle_i^{\mathsf{Op}_i(X)} y$ if and only if $x \leftarrowtriangle_i^X y$. This defines a functor $\mathsf{Op}_i(X) : \mathsf{JetSet}(\vec{a}) \to \mathsf{JetSet}(\vec{a})$.

We have $\mathsf{Op}_i \circ \mathsf{Op}_i = \mathrm{Id}$ and if $a_i = \bigcirc$ then $\mathsf{Op}_i = \mathrm{Id}$.

Definition 1.2.1°8. Write $\vec{a} \sqsubset_i \vec{b}$ if $\mathsf{len}(\vec{a}) = \mathsf{len}(\vec{b})$, $a_j = b_j$ for all $j \neq i$, $a_i = \bigcirc$ and $b_i = \nearrow$.

If $\vec{a} \sqsubset_i \vec{b}$, then we write $\mathsf{USym}_i : \mathsf{JetSet}(\vec{a}) \to \mathsf{JetSet}(\vec{b})$ for the forgetful functor which forgets the symmetry of $\rightarrowtriangle_i$.

Proposition 1.2.1°9. The forgetful functor USym_i is part of an adjoint triple $\mathsf{FSym}_i \dashv \mathsf{USym}_i \dashv \mathsf{CofSym}_i$ where FSym_i and CofSym_i take a $\vec{b}$-jet-set X to a $\vec{a}$-jet-set of the same carrier with the same j-jet relations for $j \neq i$ but where

- $x \frown_i^{\mathsf{FSym}_i X} y$ if and only if $x \rightleftarrows_i^X y$,
- $x \frown_i^{\mathsf{CofSym}_i X} y$ if and only if $x \leftrightarrowtriangle_i^X y$.

We have $\mathsf{FSym}_i \circ \mathsf{USym}_i = \mathsf{CofSym}_i \circ \mathsf{USym}_i = \mathrm{Id}$, so that $\mathsf{SymCl}_i := \mathsf{USym}_i \circ \mathsf{FSym}_i$ is an idempotent monad and $\mathsf{SymInt}_i := \mathsf{USym}_i \circ \mathsf{CofSym}_i$ is an idempotent comonad.

Definition 1.2.1°10. We extend the definition of SymCl_i and SymInt_i to endofunctors on $\mathsf{JetSet}(\vec{b})$ where b_i can be *any* anpolarity:

- If $b_i = {\times\!\!\!\nearrow}$ then they are defined as above,
- If $b_i = \bigcirc$ then they are defined as the identity functor.

Either way, they are an idempotent (co)monad and we have $\mathsf{SymCl}_i \dashv \mathsf{SymInt}_i$.

1.2.2 Intervals and Prisms

Definition 1.2.2°1. Let $\vec{a}$ be a mask and $i < \mathsf{len}(\vec{a})$.

- The i-**jet interval** $\llparenthesis \rightarrowtriangle_i \rrparenthesis$ is defined as the $\vec{a}$-jet-set with carrier $\{0, 1\}$ and relations generated by $0 \rightarrowtriangle_i 1$.
- The **opposite** i-**jet interval** $\llparenthesis \leftarrowtriangle_i \rrparenthesis$ is defined as the $\vec{a}$-jet-set with carrier $\{0, 1\}$ and relations generated by $0 \leftarrowtriangle_i 1$.
- The i-**equijet interval** $\llparenthesis \leftrightarrowtriangle_i \rrparenthesis$ is defined as the $\vec{a}$-jet-set with carrier $\{0, 1\}$ and relations generated by $0 \leftrightarrowtriangle_i 1$.

If $a_i = \bigcirc$ then $\llparenthesis \rightarrowtriangle_i \rrparenthesis = \llparenthesis \leftarrowtriangle_i \rrparenthesis = \llparenthesis \leftrightarrowtriangle_i \rrparenthesis =: \llparenthesis \frown_i \rrparenthesis$ is called the i-**edge interval**.

Note that it would be meaningless to define an i-infrajet interval in the same way.

Definition 1.2.2°2. Let $\vec{a}$ be a mask, $i < \mathsf{len}(\vec{a})$ and $X \in \mathrm{Obj}(\mathsf{JetSet}(\vec{a}))$. We define the i-**twisted-prism** $X \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis$ on X as the $\vec{a}$-jet-set with

- carrier $X \times \{0, 1\}$,
- jet relations generated by the following requirements:
  - $(\sqcup, 0) : \mathsf{Op}_i(X) \to X \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis$ is a jet set morphism,
  - $(\sqcup, 1) : X \to X \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis$ is a jet set morphism,
  - $(x, 0) \rightarrowtriangle_i (x, 1)$ for all $x \in X$.

This defines the i-**twisted-prism functor** $\sqcup \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis : \mathsf{JetSet}(\vec{a}) \to \mathsf{JetSet}(\vec{a})$.

We define the **opposite** i-**twisted-prism** $X \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis$ on X as the jet set of mask $\vec{a}$ with

- carrier $X \times \{0, 1\}$,
- jet relations generated by the following requirements:
  - $(\sqcup, 0) : X \to X \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis$ is a jet set morphism,
  - $(\sqcup, 1) : \mathsf{Op}_i(X) \to X \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis$ is a jet set morphism,
  - $(x, 0) \leftarrowtriangle_i (x, 1)$ for all $x \in X$.

This defines the **opposite** i-**twisted-prism functor** $\sqcup \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis : \mathsf{JetSet}(\vec{a}) \to \mathsf{JetSet}(\vec{a})$.

If $a_i = \bigcirc$, then we call this simply the i-**prism functor** $\sqcup \ltimes \llparenthesis \frown_i \rrparenthesis$.

Note that in both instances, we take the opposite at the source-side of the jet interval by which we multiply. This makes it unclear what a prism functor for the equijet interval $\llparenthesis \leftrightarrowtriangle_i \rrparenthesis$ should look like.

Corollary 1.2.2°3. We have $\sqcup \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis = \mathsf{Op}_i(\sqcup \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis)$. $\square$

Corollary 1.2.2°4. Let F_i be a functor between jet set categories of any of the following forms: Op_i, FSym_i, USym_i, CofSym_i, $\sqcup \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis$, $\sqcup \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis$, $\sqcup \ltimes \llparenthesis \frown_i \rrparenthesis$. Let G_j be a functor between jet set categories also of one of these forms, but for a different degree j. Then F_i and G_j commute, i.e. there is a natural isomorphism $F_i G_j \cong G_j F_i$.

Corollary 1.2.2°5. The functor $\sqcup \ltimes \llparenthesis \frown_i \rrparenthesis$ commutes with itself, i.e. $(x, v, w) \mapsto (x, w, v)$ is a natural automorphism of $\sqcup \ltimes \llparenthesis \frown_i \rrparenthesis \ltimes \llparenthesis \frown_i \rrparenthesis$. □

1.3 Jet Jewels

Write $\rightarrowtriangle_i^*$ for the transitive closure of the $\rightarrowtriangle_i$, and similarly $\rightleftarrows_i^*$ for the transitive closure of $\rightleftarrows_i$. We say that x and y are i-**connected** if $x \rightleftarrows_i^* y$, and correspondingly define i-**connected components**.

Definition 1.3.0°1. Let $\vec{a}$ be a mask of length n. An $\vec{a}$-jet-set X is a **jet jewel** if:

1. it is $(n-1)$-connected, i.e. for any $x, y \in X$ we have $x \rightleftarrows_{n-1}^* y$,
2. for every $0 \leq i < n$, the relation $\rightarrowtriangle_i^*$ is a total pre-order on any connected component of X, i.e. if $x \rightleftarrows_i^* y$, then $x \rightarrowtriangle_i^* y$ or $x \leftarrowtriangle_i^* y$. We call this property i-**orientability**.

The **category of jet jewels** $\mathsf{JetJewel}(\vec{a})$ is defined as the full subcategory of $\mathsf{JetSet}(\vec{a})$ on jet jewels.

Note that i-orientability is vacuous if $a_i = \bigcirc$.

Proposition 1.3.0°2. Jet jewels are closed under the functors Op_i, FSym_i, USym_i, $\mathsf{CofSym}_{i \neq n-1}$, SymCl_i, $\mathsf{SymInt}_{i \neq n-1}$, $\sqcup \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis$ and $\sqcup \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis$.

Proof. Op_i Trivial.

FSym_i This functor clearly preserves $(n-1)$-connectedness even if $i = n-1$. Meanwhile, j-orientability is unaffected for $j \neq i$ and vacuous for i which is symmetric in the codomain of FSym_i.

USym_i This functor does not modify the jet set, so the conditions are preserved.

$\mathsf{CofSym}_{i \neq n-1}$ Since $i \neq n-1$, this functor preserves $(n-1)$-connectedness. Meanwhile, j-orientability is unaffected for $j \neq i$ and vacuous for i which is symmetric in the codomain of CofSym_i.

SymCl_i Since $\mathsf{SymCl}_i = \mathsf{USym}_i \circ \mathsf{FSym}_i$.

$\mathsf{SymInt}_{i \neq n-1}$ Since $\mathsf{SymInt}_i = \mathsf{USym}_i \circ \mathsf{CofSym}_i$.

$\sqcup \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis$ To see $(n-1)$-connectedness, note that:

- All objects of the form $(x, 0)$ are $(n-1)$-connected,
- All objects of the form $(x, 1)$ are $(n-1)$-connected,
- Since $i \leq n-1$, we always have $(x, 0) \rightleftarrows_i (x, 1)$.

For j-orientability, we consider 3 situations:

$j < i$ Then for every x, we have $(x, 0) \not\rightleftarrows_j (x, 1)$. Every j-connected component $C \subseteq X$ produces two j-connected components $\{(c, 0) \mid c \in C\}$ and $\{(c, 1) \mid c \in C\}$, where totality of $\rightarrowtriangle_j^*$ is inherited from X.

$j > i$ Then for every x, we have $(x, 0) \leftrightarrowtriangle_j (x, 1)$. Every j-connected component $C \subseteq X$ produces one j-connected component $\{(c, u) \mid c \in C, u \in \{0, 1\}\}$, where totality of $\rightarrowtriangle_j^*$ is inherited from X.

$j = i$ Then $(x, u) \rightleftarrows_i^* (y, v)$ if and only if $x \rightleftarrows_i^* y$. By i-orientability of the original jet jewel X, we have $x \rightarrowtriangle_i^* y$ or $x \leftarrowtriangle_i^* y$. Say we have $x \rightarrowtriangle_i^* y$. Then we have $(y, 0) \rightarrowtriangle_i^* (x, 0) \rightarrowtriangle_i (x, 1) \rightarrowtriangle_i^* (y, 1)$, and thus either $(x, u) \rightarrowtriangle_i^* (y, v)$ or $(x, u) \leftarrowtriangle_i^* (y, v)$ depending on u and v.

$\sqcup \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis$ By an analogous argument. □

1.4 Jet Cubes

In section 1.4.1, we define a family of *cube* categories $\mathsf{Cube}^{⌑}_M$ parametrized by a monad M which defines the available operations on cube dimensions, and a choice $⌑ \in \{\square, \boxslash\}$ of whether we want our cubes to be affine (no diagonals) or cartesian.

In section 1.4.2, we define a family of *jet cube* categories $\mathsf{JetCube}^{⌑}_M(\omega, \vec{a})$ furthermore parametrized by a mask $\vec{a}$ and an orientation set ω, which determines whether jets can only be forward or also backward and/or bidirectional.

In section 1.4.3, we characterize morphisms in certain jet cube categories using a calculus.

In section 1.4.4, we shed light on jet cubes from a different corner by introducing the *semisymmetric separated product*.

In section 1.4.5 we compare our general class of jet cube categories to existing cube categories in the literature.

1.4.1 Cube Categories

We introduce a family of cube categories with one flavour of dimension. Fix a monad M on Set.

Example 1.4.1°1. Typically M will be one of the following:

- The 'exception' monad Pt_2 that sends a set X to $X \uplus \{0, 1\}$, which is the carrier of the free bipointed set over X;
- The monad IPt_2 that sends a set X to $X \uplus \{\neg x \mid x \in X\} \uplus \{0, 1\}$, which is the carrier of the free [bipointed set equipped with an involution $\neg$ that swaps 0 and 1] over X;
- The monad DL that sends a set X to the carrier of the free distributive lattice over X;
- The monad DM that sends a set X to the carrier of the free De Morgan algebra over X;
- The monad Boo that sends a set X to the carrier of the free boolean algebra over X.

1.4.1 (a) Cartesian Cubes

Definition 1.4.1°2. We construct the **(named) category of cartesian M-cubes** $\mathsf{Cube}^{\boxslash}_M$ (and $\mathsf{NCube}^{\boxslash}_M$ resp.) stepwise:

- The Kleisli category $\mathsf{Kl}(M)$ of M has objects $\overline{X}$ where X is a set, and its morphisms $\overline{f} : \overline{X} \to \overline{Y}$ are functions $f : X \to MY$.
- Of this, we take the opposite $\mathsf{Kl}(M)^{\mathsf{op}}$. (This is the Lawvere theory corresponding to the monad M.)
- We define $\mathsf{NCube}^{\boxslash}_M$ as the full subcategory of $\mathsf{Kl}(M)^{\mathsf{op}}$ on finite sets. (Alternatively, this is the opposite Kleisli category of the restriction of M to finite sets, either as a monad on FinSet or as a relative monad $\mathsf{FinSet} \to \mathsf{Set}$.)
- We define $\mathsf{Cube}^{\boxslash}_M$ as a designate skeleton of NCube_M, e.g. the full subcategory of $\mathsf{NCube}^{\boxslash}_M$ on sets of the form $\{0, \dots, n-1\}$ with $n \geq 0$.

Objects of $\mathsf{NCube}^{\boxslash}_M$ will be denoted as tuples of names $(\mathbf{i}_0 : \mathbb{I}, \dots, \mathbf{i}_{n-1} : \mathbb{I})$ where $\mathbb{I}$ is meaningless but conveys the intuition that we regard $\mathbf{i}_k$ as a value ranging over the interval (the cube given by the singleton object). A morphism $\varphi : (\mathbf{i}_0 : \mathbb{I}, \dots, \mathbf{i}_{n-1} : \mathbb{I}) \to (\mathbf{j}_0 : \mathbb{I}, \dots, \mathbf{j}_{m-1} : \mathbb{I})$ is then a function sending each $\mathbf{j}_k$ to an expression $\mathbf{j}_k\langle\varphi\rangle \in M\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}$. The morphism φ will also be denoted as $(\mathbf{j}_0\langle\varphi\rangle/\mathbf{j}_0, \dots, \mathbf{j}_{m-1}\langle\varphi\rangle/\mathbf{j}_{m-1})$. The situation in $\mathsf{Cube}^{\boxslash}_M$ is the same except that we now regard the names $\mathbf{i}_k$ as De Bruijn indices.

Corollary 1.4.1°3. The categories $\mathsf{Cube}^{\boxslash}_M$ and $\mathsf{NCube}^{\boxslash}_M$ have finite products, given by finite coproducts of sets. $\square$

1.4.1 (b) Affine Cubes

If T is a *container* monad [Uus17], i.e. a monad whose underlying functor is a container functor [AAG05] of the form $TX = \Sigma(s : S).(P(s) \to X)$, then we define $T^{\#}X$ as the set of *affine* expressions $\Sigma(s : S).(P(s) \hookrightarrow X)$, which is an endofunctor on the category $\mathsf{Set}^{\hookrightarrow}$ of sets and injective functions. If M is merely a *quotient* of a container monad, i.e. M is of the form $MX = TX/\sim_X$ with T as above, then we define $M^{\#}X$ as the set of equivalence classes with an affine representant.

Remark 1.4.1°4. An important source of monads such as M are monads specified by a syntactic algebraic theory [Man12, ARVL10, Nuy22]. A syntactic algebraic theory specifies a set of operations S_0, assigns to each operation $s : S_0$ an arity $P_0(s)$ which is again a set, and subjects these to a set of axioms.[3] The container (S_0, P_0) specifies a container functor $FX = \Sigma(s : S_0).(P_0(s) \to X)$ on Set. A free monad F^* over this functor F exists and satisfies the fixpoint equation $F^*X \cong X \uplus FF^*X$. We remark that the free monad F^* over a container functor F is again a container functor, i.e. there exists a container (S, P) such that $F^*X = \Sigma(s : S).(P(s) \to X)$ specifies the free monad over F. The axioms determine an equivalence relation $\sim_X$ on F^*X such that $MX := F^*X/\sim_X$ is again a monad. This situation applies to each of the monads in example 1.4.1°1.

In fact, often the quotient can be taken already at the level of the container, so that there exists a container (S', P') such that $MX \cong \Sigma(s : S').(P'(s) \to X)$.

We say that $(s, f), (s', f') \in T^{\#}X$ are **mutually fresh**, denoted $(s, f) \# (s', f')$, if the images of f and f' are disjoint. Elements of $M^{\#}X$ are mutually fresh if they have mutually fresh representants. We call the monad (T, η, μ) **affine** if $\eta_X : X \to TX$ lands in $T^{\#}X$ for all X and $\mu_X : TTX \to TX$ restricted to $(TT)^{\#}X$ (note that container functors are closed under composition) lands in $T^{\#}X$; and similar for M.

Definition 1.4.1°5. Let M be a quotient of a container monad, and let it be affine. We construct the **(named) category of affine M-cubes** $\mathsf{Cube}^{\square}_M$ (and $\mathsf{NCube}^{\square}_M$ resp.) stepwise:

- The affine Kleisli category $\mathsf{Kl}^{\#}(M)$ has objects $\overline{X}$ where X is a set, and its morphisms $\overline{f} : \overline{X} \to \overline{Y}$ are functions $f : X \to M^{\#}Y$ such that for any $x \neq x'$ in X, we have $f(x) \# f(x')$. Identity and composition are well-defined because M is affine.
- Of this, we take the opposite $\mathsf{Kl}^{\#}(M)^{\mathsf{op}}$.
- We define $\mathsf{NCube}^{\square}_M$ as the full subcategory of $\mathsf{Kl}^{\#}(M)^{\mathsf{op}}$ on finite sets.
- We define $\mathsf{Cube}^{\square}_M$ as a designate skeleton of $\mathsf{NCube}^{\square}_M$, e.g. the full subcategory of $\mathsf{NCube}^{\square}_M$ on sets of the form $\{0, \dots, n-1\}$ with $n \geq 0$.

Objects will be represented as for the cartesian cube categories.

Corollary 1.4.1°6. The categories $\mathsf{Cube}^{\square}_M$ and $\mathsf{NCube}^{\square}_M$ have a symmetric monoidal structure $(\top, *)$ given by finite coproducts of sets. The binary operation is called the **separated product**. □

1.4.1 (c) Examples

This way, we get – among others – the following cube categories:

$\mathsf{Cube}^{\boxslash}_{\mathsf{Pt}_2}$ The cartesian cube category. A morphism $\varphi : V \to W$ sends every dimension $\mathbf{j} \in W$ to $\mathbf{j}\langle\varphi\rangle \in V \cup \{0, 1\}$. Its cubes have diagonals.

$\mathsf{Cube}^{\square}_{\mathsf{Pt}_2}$ The affine cube category [BCH14]. A morphism $\varphi : V \to W$ sends every dimension $\mathbf{j} \in W$ to $\mathbf{j}\langle\varphi\rangle \in V \cup \{0, 1\}$, such that if $\mathbf{j}\langle\varphi\rangle = \mathbf{j}'\langle\varphi\rangle \in V$ then $\mathbf{j} = \mathbf{j}'$. Its cubes have no diagonals.

$\mathsf{Cube}^{\boxslash}_{\mathsf{IPt}_2}$ The symmetric cartesian cube category. We have a negation/involution/symmetry $(\neg\mathbf{i}/\mathbf{j}) : (\mathbf{i} : \mathbb{I}) \to (\mathbf{j} : \mathbb{I})$.

$\mathsf{Cube}^{\boxslash}_{\mathsf{DL}}$ The cartesian cube category with connections. We have morphisms $(\mathbf{i} \vee \mathbf{j}/\mathbf{k}), (\mathbf{i} \wedge \mathbf{j}/\mathbf{k}) : (\mathbf{i} : \mathbb{I}, \mathbf{j} : \mathbb{I}) \to (\mathbf{k} : \mathbb{I})$. There are no symmetries

[3] We use 'syntactic algebraic theory' to refer to the syntactic presentation as described here, and 'monad' and 'Lawvere theory' to refer to the less syntactic objects they specify.

$\mathsf{Cube}^{\boxslash}_{\mathsf{DM}}$ The CCHM cube category [CCHM15], which combines symmetries and connections. We have $(\mathbf{i} \wedge \neg\mathbf{i}/\mathbf{j}) \neq (0/\mathbf{j}) : (\mathbf{i} : \mathbb{I}) \to (\mathbf{j} : \mathbb{I})$ and $(\mathbf{i} \vee \neg\mathbf{i}/\mathbf{j}) \neq (1/\mathbf{j}) : (\mathbf{i} : \mathbb{I}) \to (\mathbf{j} : \mathbb{I})$.

$\mathsf{Cube}^{\boxslash}_{\mathsf{Boo}}$ A cube category very similar to the CCHM one, but we have $(\mathbf{i} \wedge \neg\mathbf{i}/\mathbf{j}) = (0/\mathbf{j}) : (\mathbf{i} : \mathbb{I}) \to (\mathbf{j} : \mathbb{I})$ and $(\mathbf{i} \vee \neg\mathbf{i}/\mathbf{j}) = (1/\mathbf{j}) : (\mathbf{i} : \mathbb{I}) \to (\mathbf{j} : \mathbb{I})$.

We remark that $\mathsf{Cube}^{\square}_{\mathsf{DM}}$ and $\mathsf{Cube}^{\square}_{\mathsf{Boo}}$ should be isomorphic as the additional law of boolean algebras w.r.t. de Morgan algebras only affects non-affine expressions.

1.4.1 (d) The Endpoint Model

We remarked above that $\mathcal{L} := \mathsf{Kl}(M)^{\mathsf{op}}$ is the Lawvere category of M. It is known then (see e.g. [Nuy22]), that the Eilenberg-Moore category of M (which is the category of Eilenberg-Moore algebras of M) is equivalent category of models of $\mathcal{L}$ (which is the category of product-preserving functors $\mathcal{L} \to \mathsf{Set}$). Such functors are fully determined by the image of the singleton set (as every set is a coproduct of singletons and the Kleisli-category retains coproducts) and that image will be exactly the carrier of the corresponding Eilenberg-Moore algebra.

It is clear that both the cartesian and affine (named) cube categories are subcategories of $\mathcal{L}$. As such, any M-algebra induces a functor $\mathcal{L} \to \mathsf{Set}$ and hence a functor from either of the M-cube categories to Set.

The initial algebra of any monad M on Set has carrier $M\varnothing$, which for each of the monads in example 1.4.1°1 equals $\{0, 1\}$. Correspondingly, the initial model of $\mathcal{L}$ is the functor $\mathsf{EP} : \mathcal{L} \to \mathsf{Set}$ sending $(\mathbf{i}_0 : \mathbb{I}, \dots, \mathbf{i}_{n-1} : \mathbb{I})$ to $\{0,1\}^{\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}}$. We call this the **endpoint model**. It is naturally isomorphic (in fact equal) to $\mathrm{Hom}_{\mathcal{L}}((), \sqcup) : \mathcal{L} \to \mathsf{Set}$, since we have

$$\mathrm{Hom}_{\mathcal{L}}((), (\mathbf{i}_0 : \mathbb{I}, \dots, \mathbf{i}_{n-1} : \mathbb{I})) = \mathrm{Hom}_{\mathsf{Kl}(M)}(\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}, \varnothing) = (M\varnothing)^{\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}} = \{0,1\}^{\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}}.$$

Recall that a morphism $\varphi : (\mathbf{i}_0 : \mathbb{I}, \dots, \mathbf{i}_{n-1} : \mathbb{I}) \to (\mathbf{j}_0 : \mathbb{I}, \dots, \mathbf{j}_{m-1} : \mathbb{I})$ assigns to each $\mathbf{j}$ a value $\mathbf{j}\langle\varphi\rangle$ in $M\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}$, the free M-algebra over $\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}$. The function $\mathsf{EP}(\varphi)$ is defined by

$$\mathsf{EP}(\varphi)\left(v^{\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\} \to \{0,1\}}\right)(\mathbf{j}) = \alpha(M(v)(\mathbf{j}\langle\varphi\rangle)),$$

where $\alpha : M\{0,1\} \to \{0,1\}$ is the algebra structure on $\{0,1\}$. Using the operation $\ggg\!=^{\alpha} : MX \to (X \to \{0,1\}) \to \{0,1\} : \hat{x} \mapsto f \mapsto \alpha(Mf(\hat{x}))$, we can write this as $\mathsf{EP}(\varphi)(v)(\mathbf{j}) = \mathbf{j}\langle\varphi\rangle \ggg\!=^{\alpha} v$.

Proposition 1.4.1°7. The functor $\mathsf{EP} : \mathsf{Cube}^{\boxslash}_{\mathsf{Boo}} \to \mathsf{Set}$ is fully faithful.

Proof. We need to show that any function $f : \{0,1\}^{\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}} \to \{0,1\}^{\{\mathbf{j}_0, \dots, \mathbf{j}_{m-1}\}}$ can be obtained as some $\mathsf{EP}(\varphi)$ with $\varphi : (\mathbf{i}_0 : \mathbb{I}, \dots, \mathbf{i}_{n-1} : \mathbb{I}) \to (\mathbf{j}_0 : \mathbb{I}, \dots, \mathbf{j}_{m-1} : \mathbb{I})$. We remark that such a function f in fact consists of m truth tables in n boolean variables. From the full disjunctive normal form, it is clear that elements of the free boolean algebra are in 1-1 correspondence with truth tables. Concretely, for each $\mathbf{j}$, define $\mathbf{j}\langle\varphi\rangle$ to be the element of $\mathsf{Boo}\{\mathbf{i}_0, \dots, \mathbf{i}_{n-1}\}$ corresponding to the truth table $f(\sqcup, \mathbf{j})$. Then $\mathbf{j}\langle\varphi\rangle \ggg\!=^{\alpha} v$ will evaluate $\mathbf{j}\langle\varphi\rangle$ after replacing each variable $\mathbf{i}$ with its value $v(\mathbf{i})$, yielding the value $f(v, \mathbf{j})$ prescribed by the truth table $f(\sqcup, \mathbf{j})$. $\square$

Proposition 1.4.1°8. The obvious functor $I : \mathsf{Cube}^{\maltese}_{M} \to \mathsf{Cube}^{⌑}_{N}$ where

- $\maltese, ⌑ \in \{\square, \boxslash\}$ and $\maltese \leq ⌑$ according to the order $\square \leq \boxslash$,
- $M, N \in \{\mathsf{Pt}_2, \mathsf{IPt}_2, \mathsf{DL}, \mathsf{Boo}\}$ and $M \leq N$ according to the partial order

$$\begin{tikzcd}
\mathsf{Pt}_2 \arrow[r, "\leq"] \arrow[d, "\leq"'] & \mathsf{IPt}_2 \arrow[d, "\leq"'] \arrow[ddr, bend left, "\leq"] & \\
\mathsf{DL} \arrow[r, "\leq"] \arrow[drr, bend right, "\leq"'] & \mathsf{DM} & \\
 & & \mathsf{Boo},
\end{tikzcd}$$

is faithful.

Proof. In a first step, it is obvious by construction that $\mathsf{Cube}^{\square}_{M} \to \mathsf{Cube}^{\boxslash}_{M}$ is faithful.

In a second step, note that we have a monad morphism $\iota : M \to N$ such that $\iota_X : M(X) \to N(X)$ is injective for all X. Then the resulting functor between the Kleisli categories, which are opposite to the cartesian cube categories, is faithful. $\square$

Clearly, the functor $I : \mathsf{Cube}^{\boxslash}_{\mathsf{DM}} \to \mathsf{Cube}^{\boxslash}_{\mathsf{Boo}}$ is not faithful: it sends the morphisms $(0/\mathbf{j}), (\mathbf{i} \wedge \neg\mathbf{i}/\mathbf{j}) : (\mathbf{i} : \mathbb{I}) \to (\mathbf{j} : \mathbb{I})$ in $\mathsf{Cube}^{\boxslash}_{\mathsf{DM}}$ to the same morphism in $\mathsf{Cube}^{\boxslash}_{\mathsf{Boo}}$.

Corollary 1.4.1°9. The functor $\mathsf{EP} : \mathsf{Cube}^{\boxtimes}_{M} \to \mathsf{Set}$ is faithful for each $M \in \{\mathsf{Pt}_2, \mathsf{IPt}_2, \mathsf{DL}, \mathsf{Boo}\}$.

Proof. Follows by composing proposition 1.4.1°7 and proposition 1.4.1°8. $\square$

1.4.2 Jet Cubes

1.4.2 (a) Jet Cube Objects

Definition 1.4.2°1. An **orientation set** is one of the following sets of formal symbols:[4]

$$\mathsf{f} = \{\rightarrowtriangle\}, \qquad \mathsf{fe} = \{\rightarrowtriangle, \leftrightarrowtriangle\}, \qquad \mathsf{fb} = \{\rightarrowtriangle, \leftarrowtriangle\}, \qquad \mathsf{fbe} = \{\rightarrowtriangle, \leftarrowtriangle, \leftrightarrowtriangle\}.$$

The set of orientation sets will be denoted $\mathbb{O} = \{\mathsf{f}, \mathsf{fe}, \mathsf{fb}, \mathsf{fbe}\}$.

Definition 1.4.2°2. Let $\vec{a}$ be a mask and ω an orientation set. We define the set of $(\omega, \vec{a})$**-jet-cubes** as the set of lists of elements of $\{P_i \mid P \in \omega, 0 \leq i < \mathsf{len}(\vec{a})\}$, where we identify all $P_i = Q_i =: \frown_i$ $(P, Q \in \omega)$ if $a_i = \bigcirc$. We denote jet cubes as $(\mathbf{i}_0 : \llparenthesis (P_0)_{i_0} \rrparenthesis, \ldots, \mathbf{i}_{n-1} : \llparenthesis (P_{n-1})_{i_{n-1}} \rrparenthesis)$, thinking of the names $\mathbf{i}_k$ as De Bruijn indices.

Definition 1.4.2°3. We call a variable $\mathbf{i}$ of an $(\omega, \vec{a})$-jet-cube i-**symmetric** (for a degree $0 \leq i < \mathsf{len}(\vec{a})$) if *any* of the following conditions holds:

- $\mathbf{i}$ is not of degree i,
- $\mathbf{i}$ is an equijet variable, i.e. $\mathbf{i} : \llparenthesis \leftrightarrowtriangle_j \rrparenthesis$,
- $a_i = \bigcirc$.

Otherwise, it is called i-**directed**. Thus, if $\mathbf{i}$ is i-directed, then $a_i = \nearrow$ and $\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis$ or $\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis$.

Definition 1.4.2°4. Let $\vec{a} \sqsubset_i \vec{b}$ and $\leftrightarrowtriangle \in \omega$. For any $(\omega, \vec{a})$-jet-cube W, we define the $(\omega, \vec{b})$-jet-cube $\mathsf{USym}^{\boxdot}_{i} W$ by replacing every occurrence of $\frown_i$ with $\leftrightarrowtriangle_i$.

Note that a $\vec{b}$-jet-cube is uniquely in the image of $\mathsf{USym}^{\boxdot}_{i}$ if it does not feature the symbols $\rightarrowtriangle_i$ and $\leftarrowtriangle_i$, i.e. if all variables are i-symmetric.

Definition 1.4.2°5. For any $(\omega, \vec{a})$-jet-cube W, we define the $\vec{a}$-jet-set $\mathsf{JEP}(W)$ as follows:

$$\begin{array}{lcll}
\mathsf{JEP}(()) & = & \top, & \\
\mathsf{JEP}(W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) & = & \mathsf{JEP}(W) \ltimes \llparenthesis \rightarrowtriangle_i \rrparenthesis, & \\
\mathsf{JEP}(W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis) & = & \mathsf{JEP}(W) \ltimes \llparenthesis \leftarrowtriangle_i \rrparenthesis, & \\
\mathsf{JEP}(W, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis) & = & \mathsf{JEP}(W) \ltimes \llparenthesis \frown_i \rrparenthesis, & \\
\mathsf{JEP}(W, \mathbf{i} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis) & = & \mathsf{USym}_i\, \mathsf{JEP}\big((\mathsf{USym}^{\boxdot}_{i})^{-1}(W, \mathbf{i} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis)\big) & \text{if } a_i = \nearrow \\
& = & \mathsf{USym}_i\, \mathsf{JEP}\big((\mathsf{USym}^{\boxdot}_{i})^{-1}(W), \mathbf{i} : \llparenthesis \frown_i \rrparenthesis\big) & \\
& = & \mathsf{USym}_i\big(\mathsf{JEP}\big((\mathsf{USym}^{\boxdot}_{i})^{-1}(W)\big) \ltimes \llparenthesis \frown_i \rrparenthesis\big). &
\end{array}$$

Setting $V = (\mathsf{USym}^{\boxdot}_{i})^{-1}(W)$, the last equation can be rephrased as

$$\begin{array}{rcl}
\mathsf{JEP}(\mathsf{USym}^{\boxdot}_{i} V, \mathbf{i} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis) = \mathsf{JEP}(\mathsf{USym}^{\boxdot}_{i}(V, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis)) & = & \mathsf{USym}_i(\mathsf{JEP}(V, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis)) \\
& = & \mathsf{USym}_i(\mathsf{JEP}(V) \ltimes \llparenthesis \frown_i \rrparenthesis).
\end{array}$$

[4] The letters stand for *forward*, *backward* and *equijet*.

Definition 1.4.2°6. We define the **jet-erasure function** $\lfloor \sqcup \rfloor$, which sends $(\omega, \vec{a})$-jet-cubes to cubes (i.e. objects of any of the cube categories defined in section 1.4.1), by

$$\lfloor () \rfloor = (), \qquad \lfloor (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \rfloor = \lfloor (W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis) \rfloor = \lfloor (W, \mathbf{i} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis) \rfloor = \lfloor (W, \mathbf{i} : \llparenthesis \frown_i \rrparenthesis) \rfloor = (\lfloor W \rfloor, i : \mathbb{I}).$$

Corollary 1.4.2°7. For any $(\omega, \vec{a})$-jet-cube W, the carrier of $\mathsf{JEP}(W)$ is $\mathsf{EP}(\lfloor W \rfloor)$. Thus, every jet cube determines an object of the following strict pullback of categories:

$$\begin{tikzcd}
\{(\omega, \vec{a})\text{-jet-cubes}\} \arrow[dr, dotted] \arrow[drr, bend left, "\mathsf{JEP}"] \arrow[ddr, bend right, "\lfloor \sqcup \rfloor"'] & & \\
& \mathsf{Cube}_M^{\text{⌘}} \times_{\mathsf{Set}} \mathsf{JetSet}(\vec{a}) \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{JetSet}(\vec{a}) \arrow[d, "U"] \\
& \mathsf{Cube}_M^{\text{⌘}} \arrow[r, "\mathsf{EP}"'] & \mathsf{Set},
\end{tikzcd}$$

It is straightforward to see that the function thus obtained is injective. $\square$

1.4.2 (b) Jet Cube Categories

Definition 1.4.2°8. Let $\vec{a}$ be a mask, $\omega \in \mathbb{O}$, $\text{⌘} \in \{\Box, \boxslash\}$ and M a monad on Set. We define the category $\mathsf{JetCube}_M^{\text{⌘}}(\omega, \vec{a})$ of $(\omega, \vec{a})$**-jet-M-cubes** as the full subcategory of $\mathsf{Cube}_M^{\text{⌘}} \times_{\mathsf{Set}} \mathsf{JetSet}(\vec{a})$ on $(\omega, \vec{a})$-jet-cubes, as justified by corollary 1.4.2°7. The functions JEP and $\lfloor \sqcup \rfloor$ are correspondingly extended to functors.

Corollary 1.4.2°9. The functor $\mathsf{JEP} : \mathsf{JetCube}_M^{\text{⌘}}(\omega, \vec{a}) \to \mathsf{JetSet}(\vec{a})$ factors over the inclusion $\mathsf{JetJewel}(\vec{a}) \hookrightarrow \mathsf{JetSet}(\vec{a})$.

Proof. By induction on the dimension and using proposition 1.3.0°2, it is clear that for any jet cube W, the jet set $\mathsf{JEP}(W)$ is a jet jewel, which factors the action on objects. The action on morphisms factors because $\mathsf{JetJewel}(\vec{a})$ is a full subcategory of $\mathsf{JetSet}(\vec{a})$. $\square$

We will ultimately only be interested in $(\mathsf{f}, \vec{a})$-jet-cubes, but in section 1.4.3, we define an inductive predicate to determine whether a cube morphism $\varphi : \lfloor V \rfloor \to \lfloor W \rfloor$ is in fact a jet cube morphism $V \to W$, and this predicate's inference rules make use of $(\mathsf{fbe}, \vec{a})$-jet-cubes in their premises. The orientation sets fe and fb are only introduced for explanatory purposes: we can easily relate jet cubes with and without $\leftarrowtriangle$ (proposition 1.4.2°10), and later on[5] we will be able to relate jet cubes with and without $\leftrightarrowtriangle$ by inserting additional symmetric degrees.

Proposition 1.4.2°10. Let $(\omega, \omega') \in \{(\mathsf{f}, \mathsf{fb}), (\mathsf{fe}, \mathsf{fbe})\}$. Then the inclusion

$$\mathsf{JetCube}_M^{\text{⌘}}(\omega, \vec{a}) \hookrightarrow \mathsf{JetCube}_M^{\text{⌘}}(\omega', \vec{a}),$$

which is fully faithful by definition, is also split essentially surjective and therefore an equivalence.

Proof. One proves, by induction on the length of the jet cube, that any jet cube is equivalent to the jet cube in which every occurrence of $\llparenthesis \leftarrowtriangle_i \rrparenthesis$ is replaced with $\llparenthesis \rightarrowtriangle_i \rrparenthesis$. $\square$

Proposition 1.4.2°11. The following functors on $\vec{a}$-jet-sets lift over JEP to functors on $(\omega, \vec{a})$-jet-cubes under the following conditions:

[5] Refer

$$\begin{array}{cccll}
\mathsf{Op}_i & \text{lifts to} & \mathsf{Op}_i^{\boxdot} & \text{if} & \leftarrowtriangle \in \omega \\
\mathsf{FSym}_i & \text{lifts to} & \mathsf{FSym}_i^{\boxdot} & & \\
\mathsf{USym}_i & \text{lifts to} & \mathsf{USym}_i^{\boxdot} & \text{if} & \leftrightarrowtriangle \in \omega \\
\mathsf{SymCl}_i & \text{lifts to} & \mathsf{SymCl}_i^{\boxdot} & \text{if} & \leftrightarrowtriangle \in \omega \\
\sqcup \ltimes (\!|\rightarrowtriangle_i|\!) & \text{lifts to} & \sqcup \ltimes (\mathbf{i} : (\!|\rightarrowtriangle_i|\!)) & & \\
\sqcup \ltimes (\!|\frown_i|\!) & \text{lifts to} & \sqcup \ltimes (\mathbf{i} : (\!|\frown_i|\!)) & \text{if} & a_i = \bigcirc \\
\sqcup \ltimes (\!|\leftarrowtriangle_i|\!) & \text{lifts to} & \sqcup \ltimes (\mathbf{i} : (\!|\leftarrowtriangle_i|\!)) & \text{if} & \leftarrowtriangle \in \omega
\end{array}$$

We have $\mathsf{FSym}_i^{\boxdot} \dashv \mathsf{USym}_i^{\boxdot}$ and thus an idempotent monad $\mathsf{SymCl}_i^{\boxdot} := \mathsf{USym}_i^{\boxdot} \circ \mathsf{FSym}_i^{\boxdot}$, whose definition we extend to masks $\vec{b}$ where $b_i = \bigcirc$ as in definition 1.2.1°10.

Proof. The functors FSym_i, Op_i and USym_i have no effect on the carrier, so they certainly lift to $\mathsf{Cube}_M^{\diamond}$, hence to the pullback $\mathsf{Cube}_M^{\diamond} \times_{\mathsf{Set}} \mathsf{JetSet}(\vec{a})$.

- FSym_i lifts to jet cubes by replacing every occurrence of $\rightarrowtriangle_i$, $\leftarrowtriangle_i$ or $\leftrightarrowtriangle_i$ with $\frown_i$.
- Op_i lifts to jet cubes if $\leftarrowtriangle \in \omega$ by reversing the *last* occurrence of either $\rightarrowtriangle_i$ or $\leftarrowtriangle_i$ (corollary 1.2.2°3), if present.
- USym_i lifts to jet cubes as the operation $\mathsf{USym}_i^{\boxdot}$ already introduced in definition 1.4.2°4.

To prove $\mathsf{FSym}^{\boxdot} \dashv \mathsf{USym}^{\boxdot}$, we need to build unit and co-unit natural transformations. Since the categories of jet cubes are fully faithful subcategories of the pullback $\mathsf{Cube}_M^{\diamond} \times_{\mathsf{Set}} \mathsf{JetSet}(\vec{a})$, it suffices to build them there. They were already established in $\mathsf{JetSet}(\vec{a})$ by proposition 1.2.1°9. As they reduce to the identity unit and co-unit of $\mathrm{Id} \dashv \mathrm{Id}$ for the carriers, they trivially lift to $\mathsf{Cube}_M^{\diamond}$.

The various prism functors multiply the carrier with $\{0, 1\}$ and thus lift over EP to the affine/cartesian cube category by multiplying with $(\mathbf{i} : \mathbb{I})$. Hence, they also lift to the pullback $\mathsf{Cube}_M^{\diamond} \times_{\mathsf{Set}} \mathsf{JetSet}(\vec{a})$. Each of them lifts to jet cubes by appending the symbol concerned (if available). □

Proposition 1.4.2°12. Any two functors on jet cubes concerned in proposition 1.4.2°11, instantiated on different degrees, commute. In other words, the natural transformation given in corollary 1.2.2°4 lifts to jet cubes when the associated functors lift.

Proof. Since the categories of jet cubes are fully faithful subcategories of the pullback $\mathsf{Cube}_M^{\diamond} \times_{\mathsf{Set}} \mathsf{JetSet}(\vec{a})$, it suffices to prove the natural isomorphism there. The isomorphism was already established in $\mathsf{JetSet}(\vec{a})$ by corollary 1.2.2°4, and the effect on the carrier is either nothing (when at most one prism functor is involved) or swapping components (when both functors are prism functors). These isomorphisms lift to $\mathsf{Cube}_M^{\diamond}$. □

Proposition 1.4.2°13. The functor $\sqcup \ltimes (\mathbf{i} : (\!|\frown_i|\!))$ commutes with itself, i.e. the natural automorphism given in corollary 1.2.2°5 lifts to jet cubes as $(\mathbf{i}/\mathbf{i}, \mathbf{j}/\mathbf{j}) : \sqcup \ltimes (\mathbf{i} : (\!|\frown_i|\!)) \ltimes (\mathbf{j} : (\!|\frown_i|\!)) \cong \sqcup \ltimes (\mathbf{j} : (\!|\frown_i|\!)) \ltimes (\mathbf{i} : (\!|\frown_i|\!))$.

Proof. Analogous to the proof of proposition 1.4.2°12. The isomorphism was already established in $\mathsf{JetSet}(\vec{a})$ by corollary 1.2.2°5, and the effect on the carrier is swapping components, which lifts to $\mathsf{Cube}_M^{\diamond}$. □

Remark 1.4.2°14. As of this point we will only be interested in the monads IPt_2 and Boo because:

- We need involutions in order to be able to work with the source-side of the twisted prism, ruling out Pt_2 and DL.
- We do not see any advantage of DM over Boo. In particular, we want EP to be faithful (proposition 1.4.1°8).

Theorem 1.4.2°15. Assuming decidability of the affineness predicate on cube morphisms, then for $M \in \{\mathsf{Pt}_2, \mathsf{IPt}_2\}$, we have isomorphisms of categories[6]

$$\mathsf{JetCube}_M^{\square}(\omega, \vec{a}) \cong \mathsf{JetCube}_M^{\boxslash}(\omega, \vec{a}),$$

which act as the identity on objects.

[6]Depending on the formalization, possibly even equalities.

Proof. We know that $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\omega, \vec{a})$ is a subcategory of $\mathsf{JetCube}^{\boxslash}_{\mathsf{IPt}_2}(\omega, \vec{a})$. So we need to show that any morphism in $\mathsf{JetCube}^{\boxslash}_{\mathsf{IPt}_2}(\mathsf{fbe}, \vec{a})$ is in fact affine. Take such a morphism $\hat{\varphi} : V \to W$ (write $\varphi = \lfloor \hat{\varphi} \rfloor$) and assume it is not affine. Since M only has nullary and unary operations, this means that W has dimensions $\mathbf{i}$ and $\mathbf{j}$ such that $\mathbf{i}\langle\varphi\rangle$ and $\mathbf{j}\langle\varphi\rangle$ are not mutually fresh, meaning that V has some dimension $\mathbf{k}$ such that $\mathbf{i}\langle\varphi\rangle, \mathbf{j}\langle\varphi\rangle \in \{\mathbf{k}, \neg\mathbf{k}\}$. Then $\mathsf{JEP}(\hat{\varphi})$ cannot be a jet set morphism as $\mathsf{JEP}(W)$ has no diagonals. This is a contradiction. □

Note that the situation is not so simple for Boo. For example, at symmetric degrees, $\mathsf{JetCube}^{\boxslash}_{\mathsf{Boo}}(\mathsf{f}, \vec{a})$ features the 'exclusive or' operation

$$((\mathbf{i} \vee \mathbf{j}) \wedge \neg(\mathbf{i} \wedge \mathbf{j})/\mathbf{k}) : (\mathbf{i} : \llparenthesis \frown_i \rrparenthesis, \mathbf{j} : \llparenthesis \frown_i \rrparenthesis) \to (\mathbf{k} : \llparenthesis \frown_i \rrparenthesis)$$

which cannot be constructed in $\mathsf{JetCube}^{\square}_{\mathsf{Boo}}(\mathsf{f}, \vec{a})$. More startlingly, even at directed degrees, we have operations such as the following:

$$(\mathbf{i} \wedge \mathbf{j}/\mathbf{p}, \mathbf{j} \wedge \mathbf{k}/\mathbf{q}) : (\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis, \mathbf{j} : \llparenthesis \rightarrowtriangle_i \rrparenthesis, \mathbf{k} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \to (\mathbf{p} : \llparenthesis \rightarrowtriangle_i \rrparenthesis, \mathbf{q} : \llparenthesis \rightarrowtriangle_i \rrparenthesis),$$

which collapses five consecutive points of the Hamiltonian path and is a legitimate jet cube morphism:

$$\begin{tikzcd}
(1,0,0) \arrow[rrr] & & & (1,0,1) \arrow[dl] \arrow[ddd, dotted] \\
& (0,0,0) \arrow[ul] \arrow[r, dotted] & (0,0,1) \arrow[d] & \\
& (0,1,0) \arrow[u] \arrow[r, dotted] & (0,1,1) \arrow[dr] & \\
(1,1,0) \arrow[uuu, dotted] \arrow[ur] \arrow[rrr, dotted] & & & (1,1,1)
\end{tikzcd}$$

$$\mapsdown$$

$$\begin{tikzcd}
(0,0) \arrow[rrr, equal] & & & (0,0) \arrow[dl, equal] \arrow[ddd, dotted] \\
& (0,0) \arrow[ul, equal] \arrow[r, equal, dotted] & (0,0) \arrow[d] & \\
& (0,0) \arrow[u, equal] \arrow[r, dotted] & (0,1) \arrow[dr] & \\
(1,0) \arrow[uuu, dotted] \arrow[ur] \arrow[rrr, dotted] & & & (1,1).
\end{tikzcd}$$

While the operations $\vee$ and $\wedge$ in themselves are useful in developing a base category for pro-arrow equipments in order to extract companion and conjoint squares from an arrow, cube transformations such as the one above are not assumed in the definition of pro-arrow equipments, so we wish to exclude these. For this reason, we will no longer be interested in cartesian jet-Boo-cubes. By theorem 1.4.2°15, we are also no longer interested in cartesian jet-IPt_2-cubes. In short then, by remark 1.4.2°14:

Remark 1.4.2°16. We are no longer interested in cartesian jet cubes.

1.4.3 A Calculus for Jet Cube Morphisms

In this section, we develop a calculus that inductively generates the morphisms of the category $\mathsf{JetCube}^{\Box}_{M}(\mathsf{fbe}, \vec{a})$ and therefore also those of its full subcategories $\mathsf{JetCube}^{\Box}_{M}(\omega, \vec{a})$.

Since the forgetful functor $U : \mathsf{JetSet}(\vec{a}) \to \mathsf{Set}$ is faithful, so is $\lfloor \sqcup \rfloor : \mathsf{JetCube}^{\Box}_{M}(\omega, \vec{a}) \to \mathsf{Cube}^{\Box}_{M}$. As such, we can regard 'being a morphism of jet cubes' as a proof-irrelevant property of morphisms of cubes, which we will therefore use as preterms. Our calculus will therefore feature a single judgement $\vdash \varphi : V \to W$ meaning that the morphism $\varphi : \lfloor V \rfloor \to \lfloor W \rfloor$ is in fact a morphism of jet cubes. Soundness (theorem 1.4.3°4) of the calculus will be the property that the judgement's meaning actually holds when the judgement is derivable, whereas completeness (theorem 1.4.3°26) means that the judgement is derivable when its meaning is true. We do not have to bother with an equational theory, as we can simply inherit it from $\mathsf{Cube}^{\Box}_{M}$.

Definition 1.4.3°1. We call a jet cube **conventional** if each of its dimensions has a degree equal to or lower than the previous one. We write $\mathsf{JetCubeConv}^{\Box}_{M}(\omega, \vec{a})$ for the full subcategory of $\mathsf{JetCube}^{\Box}_{M}(\omega, \vec{a})$ on conventional cubes.

Corollary 1.4.3°2. By proposition 1.4.2°12, the inclusion $\mathsf{JetCubeConv}^{\Box}_{M}(\omega, \vec{a}) \hookrightarrow \mathsf{JetCube}^{\Box}_{M}(\omega, \vec{a})$ is essentially surjective (in fact split essentially surjective by the existence of sorting algorithms) and thus an equivalence of categories. $\square$

Definition 1.4.3°3. For $M \in \{\mathsf{IPt}_2, \mathsf{Boo}\}$, any mask $\vec{a}$ and for any two objects

$$V, W \in \mathrm{Obj}(\mathsf{JetCubeConv}^{\Box}_{M}(\mathsf{fbe}, \vec{a})),$$

we define a proof-irrelevant predicate on morphisms $\varphi : \lfloor V \rfloor \to \lfloor W \rfloor$, denoted $\vdash \varphi : V \to W$, inductively generated by the inference rules in fig. 1.1.

We discuss these inference rules one by one; their soundness and completeness will be proven in theorems 1.4.3°4 and 1.4.3°26. Specializations of these rules for symmetric degrees are given in fig. 1.2, and some rules are grouped together in fig. 1.3.

The unique morphism to the terminal cube () is a jet cube morphism (TERMINAL).

We can substitute the last variable with an endpoint. If this end point is at the last dimension's source side, then the rest of the morphism lands in the i-opposite of W (SRC:FWD, SRC:BCK), otherwise it lands in W itself (TGT:FWD, TGT:BCK).

We can apply an involution to the last variable, provided that we turn around its direction (INV:FWD, INV:BCK). Doing so means that the source-side is mapped to the source-side and the target-side is mapped to the target-side, so W remains unaffected.

We can apply the (opposite) i-twisted prism functor to a morphism (PRISM:FWD, PRISM:BCK).

If the last dimension of our target cube is of the form $\mathbf{i} : (\!|\leftrightarrow_i|\!)$, then we know that our cube is in the image of $\mathsf{USym}^{\boxdot}_{i}$, and we can proceed using the adjunction $\mathsf{FSym}^{\boxdot}_{i} \dashv \mathsf{USym}^{\boxdot}_{i}$ (SYMMETRIZE). This turns our last dimension into $\mathbf{i} : (\!|\frown_i|\!)$ which is a special case of both $\mathbf{i} : (\!|\rightarrow_i|\!)$ and $\mathbf{i} : (\!|\leftarrow_i|\!)$, so we can proceed by using the FWD and BCK rules of the calculus.

We can weaken w.r.t. the last dimension (WKN) of the source cube, but some caution is required. At the source-side of the last dimension, we find $\mathsf{Op}^{\boxdot}_{i}(V)$, whereas at the target-side we have V. Thus, φ needs to be a morphism of jet cubes from $\mathsf{Op}^{\boxdot}_{i}(V) \to W$ as well as from $V \to W$. This can be achieved by asking that φ starts from $\mathsf{SymCl}^{\boxdot}_{i}(V)$, which can be thought of as a join of $\mathsf{Op}^{\boxdot}_{i}(V)$ and V. In the case of an equijet dimension, $\mathsf{SymCl}^{\boxdot}_{i}(V) = V = \mathsf{Op}^{\boxdot}_{i}(V)$.

We can exchange variables of the same symmetric degree i (EXCHANGE). Note that conventionality implies that all variables in U_1 are also of type $(\!|\leftrightarrow_i|\!)$.

We can substitute the last variable with a variable of a weaker (higher) degree in either direction (or of equijet dimension). Inspired by proposition 1.2.1°5, we choose to use terminology from pro-arrow equipments and refer to this action as creating a companion when the direction of the arrow remains the same ($P = Q$), and a conjoint when it reverses ($\{P, Q\} = \{\rightarrow, \leftarrow\}$); we introduce the term **concursor** (CONCURSOR) as the common generalization of companions, conjoints, and their symmetric counterpart

TERMINAL
$$\frac{}{\vdash () : V \to ()}$$

SRC:FWD
$$\frac{\vdash \varphi : V \to \mathsf{Op}_i^{\boxdot}(W)}{\vdash (\varphi, 0/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)}$$

SRC:BCK
$$\frac{\vdash \varphi : V \to \mathsf{Op}_i^{\boxdot}(W)}{\vdash (\varphi, 1/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis)}$$

TGT:FWD
$$\frac{\vdash \varphi : V \to W}{\vdash (\varphi, 1/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)}$$

TGT:BCK
$$\frac{\vdash \varphi : V \to W}{\vdash (\varphi, 0/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis)}$$

INV:FWD
$$\frac{\vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis)}{\vdash (\varphi, \neg t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)}$$

INV:BCK
$$\frac{\vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)}{\vdash (\varphi, \neg t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis)}$$

PRISM:FWD
$$\frac{\vdash \varphi : V \to W}{\vdash (\varphi, \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)}$$

PRISM:BCK
$$\frac{\vdash \varphi : V \to W}{\vdash (\varphi, \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis)}$$

SYMMETRIZE
$$\frac{\vdash \varphi : \mathsf{FSym}_i^{\boxdot} V \to W}{\vdash \varphi : V \to \mathsf{USym}_i^{\boxdot} W}$$

WKN
$$\frac{\vdash \varphi : \mathsf{SymCl}_i^{\boxdot}(V) \to W \qquad R \in \{\rightarrowtriangle, \leftarrowtriangle, \leftrightarrowtriangle\}}{\vdash (\varphi, \mathbf{i}/\oslash) : (V, \mathbf{i} : \llparenthesis R_i \rrparenthesis) \to W}$$

EXCHANGE
$$\frac{\vdash \varphi : (V, \mathbf{j} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis, U_1, \mathbf{i} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis, U_2) \to W}{\vdash \varphi : (V, \mathbf{i} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis, U_1, \mathbf{j} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis, U_2) \to W}$$

CONCURSOR
$$\frac{\begin{array}{l} P \in \{\rightarrowtriangle, \leftarrowtriangle, \leftrightarrowtriangle\} \qquad Q \in \{\rightarrowtriangle, \leftarrowtriangle\} \qquad j > i \\ \vdash \varphi : \mathsf{SymCl}_i^{\boxdot}(\mathsf{SymCl}_j^{\boxdot} U, V) \to W \end{array}}{\vdash (\varphi, \mathbf{j}/\mathbf{i}) : (U, \mathbf{j} : \llparenthesis P_j \rrparenthesis, V) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}$$

CONN:PRISM:SRC-NEUTRAL
$$\frac{\begin{array}{l} (Q, \diamondsuit) \in \{(\rightarrowtriangle, \vee), (\leftarrowtriangle, \wedge)\} \\ \vdash \varphi : \mathsf{SymCl}_i^{\boxdot} V \to W \\ \vdash (\varphi, t/\mathbf{i}) : \mathsf{Op}_i^{\boxdot} V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \end{array}}{\vdash (\varphi, t \diamondsuit \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\,\text{Boo}$$

CONN:PRISM:TGT-NEUTRAL
$$\frac{\begin{array}{l} (Q, \diamondsuit) \in \{(\rightarrowtriangle, \wedge), (\leftarrowtriangle, \vee)\} \\ \vdash \varphi : \mathsf{SymCl}_i^{\boxdot} V \to W \\ \vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \end{array}}{\vdash (\varphi, t \diamondsuit \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\,\text{Boo}$$

CONN:PRISM-INV:SRC-NEUTRAL
$$\frac{\begin{array}{l} (Q, \diamondsuit, P) \in \{(\rightarrowtriangle, \vee, \leftarrowtriangle), (\leftarrowtriangle, \wedge, \rightarrowtriangle)\} \\ \vdash \varphi : \mathsf{SymCl}_i^{\boxdot} V \to W \\ \vdash (\varphi, t/\mathbf{i}) : \mathsf{Op}_i^{\boxdot} V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \end{array}}{\vdash (\varphi, t \diamondsuit \neg\mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis P_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\,\text{Boo}$$

CONN:PRISM-INV:TGT-NEUTRAL
$$\frac{\begin{array}{l} (Q, \diamondsuit, P) \in \{(\rightarrowtriangle, \wedge, \leftarrowtriangle), (\leftarrowtriangle, \vee, \rightarrowtriangle)\} \\ \vdash \varphi : \mathsf{SymCl}_i^{\boxdot} V \to W \\ \vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \end{array}}{\vdash (\varphi, t \diamondsuit \neg\mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : \llparenthesis P_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\,\text{Boo}$$

CONN:DEGREE-SYMMETRIC
$$\frac{\begin{array}{l} Q \in \{\rightarrowtriangle, \leftarrowtriangle\} \qquad \diamondsuit \in \{\vee, \wedge\} \\ \vdash (\varphi, s/\mathbf{i}) : \mathsf{SymCl}_i^{\boxdot} V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \\ \vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis) \end{array}}{\vdash (\varphi, t \diamondsuit s/\mathbf{i}) : V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)}\,\text{Boo}$$

Figure 1.1: A calculus of affine fbe-jet-cube morphisms, for the monads IPt_2 and Boo. See fig. 1.2 for specializations of these rules to symmetric degrees and fig. 1.3 for unified versions of the specialized forward/backward rules.

$$\textsc{endpoint:sym}\quad \frac{\vdash \varphi : V \to W \qquad c \in \{0,1\}}{\vdash (\varphi, c/\mathbf{i}) : V \to (W, \mathbf{i} : (\!|\frown_i|\!))}$$

$$\textsc{inv:sym}\quad \frac{\vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : (\!|\frown_i|\!))}{\vdash (\varphi, \neg t/\mathbf{i}) : V \to (W, \mathbf{i} : (\!|\frown_i|\!))}$$

$$\textsc{prism:sym}\quad \frac{\vdash \varphi : V \to W}{\vdash (\varphi, \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : (\!|\frown_i|\!)) \to (W, \mathbf{i} : (\!|\frown_i|\!))}$$

$$\textsc{wkn:sym}\quad \frac{\vdash \varphi : V \to W}{\vdash (\varphi, \mathbf{i}/\oslash) : (V, \mathbf{i} : (\!|\frown_i|\!)) \to W}$$

$$\textsc{exchange:sym}\quad \frac{\vdash \varphi : (V, \mathbf{j} : (\!|\frown_i|\!), U_1, \mathbf{i} : (\!|\frown_i|\!), U_2) \to W}{\vdash \varphi : (V, \mathbf{i} : (\!|\frown_i|\!), U_1, \mathbf{j} : (\!|\frown_i|\!), U_2) \to W}$$

$$\textsc{conn:sym}\quad \frac{\begin{array}{l}\diamondsuit \in \{\vee, \wedge\}\\ \vdash (\varphi, s/\mathbf{i}) : V \to (W, \mathbf{i} : (\!|\frown_i|\!))\\ \vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : (\!|\frown_i|\!))\end{array}}{\vdash (\varphi, t \diamondsuit s/\mathbf{i}) : V \to (W, \mathbf{i} : (\!|\frown_i|\!))}\,\mathsf{Boo}$$

Figure 1.2: Symmetric specializations of the rules in fig. 1.1. Note that the rules conn:prism:* become a special case of conn:degree-symmetric. We omit terminal which specifies no degree, concursor which specifies two, and symmetrize which already places constraints on the anpolarity a_i.

$$\textsc{src}\quad \frac{\begin{array}{l}(Q, c) \in \boxed{\{(\rightarrowtriangle, 0), (\leftarrowtriangle, 1)\}}\\ \vdash \varphi : V \to \boxed{\mathsf{Op}_i^{\boxdot}(W)}\end{array}}{\vdash (\varphi, c/\mathbf{i}) : V \to (W, \mathbf{i} : (\!|Q_i|\!))}$$

$$\textsc{tgt}\quad \frac{\begin{array}{l}(Q, c) \in \boxed{\{(\rightarrowtriangle, 1), (\leftarrowtriangle, 0)\}}\\ \vdash \varphi : V \to \boxed{W}\end{array}}{\vdash (\varphi, c/\mathbf{i}) : V \to (W, \mathbf{i} : (\!|Q_i|\!))}$$

$$\textsc{inv}\quad \frac{\begin{array}{l}\{P, Q\} = \{\rightarrowtriangle, \leftarrowtriangle\}\\ \vdash (\varphi, t/\mathbf{i}) : V \to (W, \mathbf{i} : (\!|P_i|\!))\end{array}}{\vdash (\varphi, \neg t/\mathbf{i}) : V \to (W, \mathbf{i} : (\!|Q_i|\!))}$$

$$\textsc{prism}\quad \frac{\vdash \varphi : V \to W \qquad Q \in \{\rightarrowtriangle, \leftarrowtriangle\}}{\vdash (\varphi, \mathbf{i}/\mathbf{i}) : (V, \mathbf{i} : (\!|Q_i|\!)) \to (W, \mathbf{i} : (\!|Q_i|\!))}$$

Figure 1.3: Unified versions of the specialized forward/backward rules in fig. 1.1.

equiconcursors ($P = \leftrightarrowtriangle$). Some measures of caution need to be taken however, which we consider in the case of forward companions ($P = Q = \rightarrowtriangle$), where we wish to derive $(\varphi, \mathbf{j}/\mathbf{i}) : (U, \mathbf{j} : (\!|\rightarrowtriangle_j|\!), V) \to (W, \mathbf{i} : (\!|\rightarrowtriangle_i|\!))$. First of all, we need to enforce affineness and make sure that φ does not use the variable **j**, so we will have $\varphi : (\lfloor U \rfloor, \lfloor V \rfloor) \to \lfloor W \rfloor$. Now let us look at what happens when we set **i** and **j** to 0 or to 1:

$$\begin{tikzcd}
(\mathsf{Op}_j^{\boxdot}(U), V) \arrow[r, dotted, "\varphi"] \arrow[d, "(0/\mathbf{j})"'] & \mathsf{Op}_i^{\boxdot}(W) \arrow[d, "(0/\mathbf{i})"] \\
(U, \mathbf{j} : (\!|\rightarrowtriangle_j|\!), V) \arrow[r, "(\varphi, \mathbf{j}/\mathbf{i})"'] & (W, \mathbf{i} : (\!|\rightarrowtriangle_i|\!)) \\
(U, V) \arrow[r, dotted, "\varphi"'] \arrow[u, "(1/\mathbf{j})"] & W \arrow[u, "(1/\mathbf{i})"']
\end{tikzcd}$$

So φ needs to be both a morphism of jet cubes from (U, V) to W and from $(\mathsf{Op}_j^{\boxdot}(U), V)$ to $\mathsf{Op}_i^{\boxdot}(W)$ or equivalently from $\mathsf{Op}_i^{\boxdot}(\mathsf{Op}_j^{\boxdot}(U), V)$ to W. This can be achieved by asking that φ starts from $\mathsf{SymCl}_i(\mathsf{SymCl}_j(U), V)$, which can be thought of as a join of (U, V) and $\mathsf{Op}_i^{\boxdot}(\mathsf{Op}_j^{\boxdot}(U), V)$.

The last five rules involve connections (conjunction and disjunction) and only apply if $M = \mathsf{Boo}$, as IPt_2 does not provide these operations. Due to the twisted nature of the twisted prism functor, it turns out that we can substitute the last variable of the target cube only with a connection of which one operand (say the right one) is either (the negation of) the last variable of the source cube, or an expression depending only on i-symmetric variables.[7]

In either case, it turns out that whether the last term reduces to 0 or 1 on any end point of the source cube, is sufficiently irregular that i-directed terms in the remainder of φ can only depend on i-symmetric variables. The promotion of all variables of degree i to equijet variables in the context of φ in at least one of the premises, precludes their usage in i-directed terms.

In the latter case, we can use CONN:DEGREE-SYMMETRIC, where s is also checked in the symmetrized context.

In the former case, we get to apply a rule that combines a connection, possibly an inversion, and the PRISM rules. The main point worth remarking upon is that the behaviour of t only matters when the other operand reduces to the neutral element of the connection at hand. Depending on this, we decide whether t must be checked in the i-opposite context or not. For example, in CONN:PRISM:SRC-NEUTRAL, if $Q = \multimap$ and $\diamondsuit = \vee$, then the neutral element is 0, so the behaviour of t only matters when the other operand is 0. This means that we are coming from the source-side of $\mathbf{i}$, i.e. from $\mathsf{Op}_i^{\boxdot} V$. This distinction leads to four different rules (CONN:PRISM:SRC-NEUTRAL, CONN:PRISM:TGT-NEUTRAL, CONN:PRISM-INV:SRC-NEUTRAL, CONN:PRISM-INV:SRC-NEUTRAL).

It is worth pointing out that if $a_i = \bigcirc$, then CONN:DEGREE-SYMMETRIC specializes to the rule CONN:SYM (fig. 1.2) that is sufficiently general to also subsume the symmetric specializations of the other connection rules.

1.4.3 (a) Soundness

Theorem 1.4.3°4 (Soundness)**.** If a morphism $\varphi : \lfloor V \rfloor \to \lfloor W \rfloor$ satisfies the predicate $\vdash \varphi : V \to W$ from definition 1.4.3°3, then it actually arises as the image $\varphi = \lfloor \hat{\varphi} \rfloor$ of a morphism $\hat{\varphi} : V \to W$.

Remark 1.4.3°5. The derivation rules in our calculus are all natural w.r.t. V and W. Indeed, since $\lfloor \sqcup \rfloor$ is faithful, this is simply inherited from the underlying operations on cubes.

Proof. Note that what really needs to be proven is that $\vdash \varphi : V \to W$ implies that $\mathsf{EP}(\varphi)$, which a priori is a function from the set $\mathsf{EP}(\lfloor V \rfloor) = U(\mathsf{JEP}(V))$ to $\mathsf{EP}(\lfloor W \rfloor) = U(\mathsf{JEP}(W))$, is in fact a morphism of jet sets $\mathsf{JEP}(V) \to \mathsf{JEP}(W)$. We prove this, of course, by induction on the derivation of the inductive predicate.

- For TERMINAL, note that $\mathsf{JEP}(())$ is the terminal jet set.
- For SRC:FWD, SRC:BCK, TGT:FWD and TGT:BCK, this follows immediately from definition 1.2.2°2.
- For INV:FWD, by postcomposition, it suffices to show that $\zeta : (\mathrm{id}_W, \neg\mathbf{i}/\mathbf{i}) : \lfloor (W, i : (\!|\!\leftarrow_i\!|\!)) \rfloor \to \lfloor (W, i : (\!|\!\multimap_i\!|\!)) \rfloor$ is a morphism of jet cubes, i.e. that $\mathsf{EP}(\zeta) : (\vec{w}, u) \mapsto (\vec{w}, \neg u)$ is a morphism of jet sets $\mathsf{JEP}(W) \ltimes (\!|\!\leftarrow_i\!|\!) \to \mathsf{JEP}(W) \ltimes (\!|\!\multimap_i\!|\!)$.
  Let $(\vec{w}, u) \multimap_j (\vec{w}', u')$ in $\mathsf{JEP}(W) \ltimes (\!|\!\leftarrow_i\!|\!)$. Then by definition 1.2.2°2 of the opposite i-twisted prism, there are 3 possibilities:
  - We have $u = u' = 0$ and $\vec{w} \multimap_j \vec{w}'$ in $\mathsf{JEP}(W)$. In that case, we also have the required jet between the images $(\vec{w}, 1) \multimap_j (\vec{w}', 1)$ in $\mathsf{JEP}(W) \ltimes (\!|\!\multimap_i\!|\!)$.
  - We have $u = u' = 1$ and $\vec{w} \multimap_j \vec{w}'$ in $\mathsf{Op}_i(\mathsf{JEP}(W))$. In that case, we also have the required jet between the images $(\vec{w}, 0) \multimap_j (\vec{w}', 0)$ in $\mathsf{JEP}(W) \ltimes (\!|\!\multimap_i\!|\!)$.
  - We have $j = i$, $u = 1$, $u' = 0$ and $\vec{w} = \vec{w}'$. In that case, we also have the required jet between the images $(\vec{w}, 0) \multimap_i (\vec{w}, 1)$ in $\mathsf{JEP}(W) \ltimes (\!|\!\multimap_i\!|\!)$.

  The proof of soundness of INV:BCK is analogous.
- Soundness of PRISM:FWD and PRISM:BCK was already established by proposition 1.4.2°11.
- Soundness of SYMMETRIZE follows from the adjunction established in proposition 1.4.2°11.
- We prove soundness of WKN by precomposition with a jet cube morphism that erases to $(\mathrm{id}, \mathbf{i}/\oslash)$: $\lfloor (V, \mathbf{i} : (\!|R_i|\!)) \rfloor \to \mathsf{SymCl}_i \lfloor V \rfloor$. Thus, we need to prove that $\mathsf{EP}(\mathrm{id}, \mathbf{i}/\oslash) : (\vec{v}, w) \mapsto \vec{v}$ is a jet set morphism $\mathsf{JEP}(V, \mathbf{i} : (\!|R_i|\!)) \to \mathsf{JEP}(\mathsf{SymCl}_i^{\boxdot}(V))$. Let $(\vec{v}, w) \multimap_j (\vec{v}', w')$ in $\mathsf{JEP}(V, \mathbf{i} : (\!|R_i|\!))$. Then there are two possibilities:

[7] This is formalized in lemma 1.4.3°23.

  - $w = w'$ and $(\vec{v}, w) \rightarrowtriangle_j (\vec{v}', w)$. The latter implies $\vec{v} \rightarrowtriangle_j \vec{v}'$ in $\mathsf{JEP}(V)$ if $j \neq i$ and $\vec{v} \leftrightarrows_i \vec{v}'$ if $j = i$. Moving to $\mathsf{JEP}(\mathsf{SymCl}^{\boxdot}_i(V))$, we get $\vec{v} \rightarrowtriangle_j \vec{v}'$ in all cases, as required.
  - $j = i$, $\vec{v} = \vec{v}'$ and $w \rightarrowtriangle_i w'$. In this case we have $\vec{v} \rightarrowtriangle_j \vec{v}$ in $\mathsf{JEP}(\mathsf{SymCl}^{\boxdot}_i(V))$ by reflexivity.

- Recalling definition 1.4.2°5, soundness of EXCHANGE follows from proposition 1.4.2°13. (Note that, since we are dealing with conventional cubes, all variables in U_1 also have type $\llparenthesis \leftrightarrowtriangle_i \rrparenthesis$.)
- We prove soundness of CONCURSOR. Assume that φ is a jet cube morphism $\mathsf{SymCl}^{\boxdot}_i(\mathsf{SymCl}^{\boxdot}_j U, V) \to W$ and $j > i$. We prove that $(\varphi, \mathbf{j}/\mathbf{i})$ is a jet cube morphism $(U, \mathbf{j} : \llparenthesis P_j \rrparenthesis, V) \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)$. Write $f = \mathsf{EP}(\varphi)$ and $g = \mathsf{EP}(\varphi, \mathbf{j}/\mathbf{i})$.
  Pick a jet $(\vec{u}, t, \vec{v}) \rightarrowtriangle_k (\vec{u}', t', \vec{v}')$ in $\mathsf{JEP}(U, \mathbf{j} : \llparenthesis P_j \rrparenthesis, V)$. We can assume that this jet is not reflexive. Let $\mathbf{k}$ be the variable where both hands differ. There are three possibilities:
  - If $\mathbf{k} \in U$, then we have $t = t'$, $\vec{v} = \vec{v}'$ and $(\vec{u}, t, \vec{v}) \rightarrowtriangle_k (\vec{u}', t, \vec{v})$.
    * If $k \neq j$ and $k \neq i$, this implies $(\vec{u}, \vec{v}) \rightarrowtriangle_k (\vec{u}', \vec{v})$ in $\mathsf{JEP}(U, V)$ and therefore also in $\mathsf{JEP}(\mathsf{SymCl}^{\boxdot}_i(\mathsf{SymCl}^{\boxdot}_j U, V))$, whence $f(\vec{u}, \vec{v}) \rightarrowtriangle_k f(\vec{u}', \vec{v})$, whence $g(\vec{u}, t, \vec{v}) = (f(\vec{u}, \vec{v}), t) \rightarrowtriangle_k (f(\vec{u}', \vec{v}), t) = g(\vec{u}', t, \vec{v})$.
    * If $k = i$ or $k = j$, this implies $(\vec{u}, \vec{v}) \leftrightarrows_k (\vec{u}', \vec{v})$ in $\mathsf{JEP}(U, V)$ and therefore $(\vec{u}, \vec{v}) \leftrightarrowtriangle_k (\vec{u}', \vec{v})$ in $\mathsf{JEP}(\mathsf{SymCl}^{\boxdot}_i(\mathsf{SymCl}^{\boxdot}_j U, V))$, whence $f(\vec{u}, \vec{v}) \leftrightarrowtriangle_k f(\vec{u}', \vec{v})$, whence $g(\vec{u}, t, \vec{v}) = (f(\vec{u}, \vec{v}), t) \leftrightarrowtriangle_k (f(\vec{u}', \vec{v}), t) = g(\vec{u}', t, \vec{v})$.
  - If $\mathbf{k} = \mathbf{j}$, then we have $k = j$, $\vec{u} = \vec{u}'$, $t \neq t'$ and $\vec{v} = \vec{v}'$. Then $g(\vec{u}, t, \vec{v}) = (f(\vec{u}, \vec{v}), t) \leftrightarrows_i (f(\vec{u}, \vec{v}), t') = g(\vec{u}, t', \vec{v})$ which implies $g(\vec{u}, t, \vec{v}) \rightarrowtriangle_j g(\vec{u}, t', \vec{v})$ since $j > i$.
  - If $\mathbf{k} \in V$, then we have $t = t'$, $\vec{u} = \vec{u}'$ and $\vec{v} \rightarrowtriangle_k \vec{v}'$ in $\mathsf{JEP}(V)$.
    * If $k \neq i$, this implies $(\vec{u}, \vec{v}) \rightarrowtriangle_k (\vec{u}, \vec{v}')$ in $\mathsf{JEP}(\mathsf{SymCl}^{\boxdot}_i(\mathsf{SymCl}^{\boxdot}_j U, V))$, whence $f(\vec{u}, \vec{v}) \rightarrowtriangle_k f(\vec{u}, \vec{v}')$, whence $g(\vec{u}, t, \vec{v}) = (f(\vec{u}, \vec{v}), t) \rightarrowtriangle_k (f(\vec{u}, \vec{v}'), t) = g(\vec{u}, t, \vec{v}')$.
    * If $k = i$, this implies $(\vec{u}, \vec{v}) \leftrightarrowtriangle_i (\vec{u}, \vec{v}')$ in $\mathsf{JEP}(\mathsf{SymCl}^{\boxdot}_i(\mathsf{SymCl}^{\boxdot}_j U, V))$, whence $f(\vec{u}, \vec{v}) \leftrightarrowtriangle_i f(\vec{u}, \vec{v}')$, whence $g(\vec{u}, t, \vec{v}) = (f(\vec{u}, \vec{v}), t) \leftrightarrowtriangle_i (f(\vec{u}, \vec{v}'), t) = g(\vec{u}, t, \vec{v}')$.

- We prove soundness of CONN:PRISM:SRC-NEUTRAL for the case where $Q = \rightarrowtriangle$ and $\diamondsuit = \vee$, the other case is proven analogously. Assume that φ is a jet cube morphism $\mathsf{SymCl}^{\boxdot}_i V \to W$ and $(\varphi, t/\mathbf{i})$ is a jet cube morphism $\mathsf{Op}^{\boxdot}_i V \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$. We prove that $(\varphi, t \vee \mathbf{i}/\mathbf{i})$ is a jet cube morphism $(V, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$. Write $f = \mathsf{EP}(\varphi)$ and $g = \mathsf{EP}(\varphi, t \vee \mathbf{i}/\mathbf{i})$ and $h = \mathsf{EP}(\varphi, t/\mathbf{i})$.
  Pick a non-reflexive jet $(\vec{v}, u) \rightarrowtriangle_j (\vec{v}', u')$ in $\mathsf{JEP}(V, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$. Let $\mathbf{k}$ be the variable where both hands defer. There are two possibilities:
  - If $\mathbf{k} = \mathbf{i}$, then $j = i$, $\vec{v} = \vec{v}'$, $u = 0$ and $u' = 1$. In this case, $(t \vee \mathbf{i})\langle \vec{v}, 1 \rangle = 1$, so that there is necessarily an i-jet $g(\vec{v}, 0) = (f(\vec{v}), (t \vee \mathbf{i})\langle \vec{v}, 0 \rangle) \rightarrowtriangle_i (f(\vec{v}), 1) = g(\vec{v}, 1)$.
  - If $\mathbf{k} \in V$, then $u = u'$ and $(\vec{v}, u) \rightarrowtriangle_j (\vec{v}', u)$. Let $\mathbf{l}$ be the variable in $(W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$ such that $\mathbf{l}\langle \varphi, t \vee \mathbf{i} \rangle$ depends on $\mathbf{k}$. If there is no such variable, then we are done.
    * If $\mathbf{l} = \mathbf{i}$, then $\mathbf{k}$ occurs in t.
      · If $u = 1$, then $g(\vec{v}, 1) = (f(\vec{v}), 1) \overset{=}{\rightarrowtriangle}_j (f(\vec{v}'), 1) = g(\vec{v}', 1)$ as required.
      · If $u = 0$, then we have $\vec{v} \rightarrowtriangle_j \vec{v}'$ in $\mathsf{JEP}(\mathsf{Op}^{\boxdot}_i V)$. Because $(\varphi, t/\mathbf{i})$ is a jet cube morphism $\mathsf{Op}^{\boxdot}_i V \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$, we get $g(\vec{v}, 0) = h(\vec{v}) \rightarrowtriangle_j h(\vec{v}') = g(\vec{v}', 0)$ as required.
    * If $\mathbf{l} \in W$, then $\mathbf{k}$ occurs in φ. Define $z = (t \vee \mathbf{i})\langle \vec{v}, u \rangle = (t \vee \mathbf{i})\langle \vec{v}', u \rangle$. We have $g(\vec{v}, u) = (f(\vec{v}), z)$ and $g(\vec{v}', u) = (f(\vec{v}'), z)$.
      · If $j = i$, then we have $\vec{v} \leftrightarrowtriangle_i \vec{v}'$ in $\mathsf{JEP}(\mathsf{SymCl}^{\boxdot}_i V)$, whence $f(\vec{v}) \leftrightarrowtriangle_i f(\vec{v}')$ in $\mathsf{JEP}(W)$, whence $g(\vec{v}, u) = (f(\vec{v}), z) \rightarrowtriangle_i (f(\vec{v}'), z) = g(\vec{v}', u)$ in $\mathsf{JEP}(W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$.
      · If $j \neq i$, then we have $\vec{v} \rightarrowtriangle_j \vec{v}'$ in $\mathsf{JEP}(\mathsf{SymCl}^{\boxdot}_i V)$, whence $f(\vec{v}) \rightarrowtriangle_j f(\vec{v}')$ in $\mathsf{JEP}(W)$, whence $g(\vec{v}, u) = (f(\vec{v}), z) \rightarrowtriangle_j (f(\vec{v}'), z) = g(\vec{v}', u)$ in $\mathsf{JEP}(W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$.

- Soundness of CONN:PRISM:TGT-NEUTRAL is proven analogously to that of CONN:PRISM:SRC-NEUTRAL.
- Soundness of CONN:PRISM-INV:SRC-NEUTRAL is proven from soundness of CONN:PRISM:SRC-NEUTRAL by precomposing the result with $(\mathrm{id}_V, \neg\mathbf{i}/\mathbf{i})$ which is a jet cube morphism $(V, \mathbf{i} : \llparenthesis P_i \rrparenthesis) \to (V, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)$.
- Soundness of CONN:PRISM-INV:TGT-NEUTRAL is similarly proven from soundness of CONN:PRISM:TGT-NEUTRAL.

- We prove soundness of CONN:DEGREE-SYMMETRIC. Assume that
  - $(\varphi, s/\mathbf{i})$ is a jet cube morphism $\mathsf{SymCl}_i^{\boxdot} V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)$,
  - $(\varphi, t/\mathbf{i})$ is a jet cube morphism $V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)$.

  We prove that $(\varphi, s \diamondsuit t/\mathbf{i})$ is a jet cube morphism $V \to (W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)$. Write

  $$f = \mathsf{EP}(\varphi), \qquad g = \mathsf{EP}(\varphi, s/\mathbf{i}), \qquad h = \mathsf{EP}(\varphi, t/\mathbf{i}), \qquad d = \mathsf{EP}(\varphi, s \diamondsuit t/\mathbf{i}).$$

  Pick a non-reflexive jet $\vec{v} \mathrel{-\!\triangleright}_j \vec{v}'$ in $\mathsf{JEP}(V)$; we prove that $d(\vec{v}) \mathrel{-\!\triangleright} d(\vec{v}')$ in $\mathsf{JEP}(W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)$. Let $\mathbf{k}$ be the variable of V where $\vec{v}$ and $\vec{v}'$ differ. There are four possible cases:
  - If φ, s and t do not depend on $\mathbf{k}$ then the target jet is reflexive.
  - If φ or s depends on $\mathbf{k}$, then we have $t\langle\vec{v}\rangle = t\langle\vec{v}'\rangle =: t_0$.
    - If $j \neq i$, then we have $\vec{v} \mathrel{-\!\triangleright}_j \vec{v}'$ in $\mathsf{JEP}(\mathsf{SymCl}_i^{\boxdot} V)$, whence $(f(\vec{v}), s\langle\vec{v}\rangle) = g(\vec{v}) \mathrel{-\!\triangleright}_j g(\vec{v}') = (f(\vec{v}'), s\langle\vec{v}'\rangle)$ in $\mathsf{JEP}(W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)$. Taking a connection with t_0 does not influence the direction of the arrows to the left of $\mathbf{i}$, nor of the arrows at $\mathbf{i}$. Hence we get $d(\vec{v}) = (f(\vec{v}), s\langle\vec{v}\rangle \diamondsuit t_0) \mathrel{-\!\triangleright}_j (f(\vec{v}'), s\langle\vec{v}'\rangle \diamondsuit t_0) = d(\vec{v}')$.
    - If $j = i$, then we have $\vec{v} \mathrel{\triangleleft\!\triangleright}_i \vec{v}'$ in $\mathsf{JEP}(\mathsf{SymCl}_i^{\boxdot} V)$, whence $(f(\vec{v}), s\langle\vec{v}\rangle) = g(\vec{v}) \mathrel{\triangleleft\!\triangleright}_i g(\vec{v}') = (f(\vec{v}'), s\langle\vec{v}'\rangle)$ in $\mathsf{JEP}(W, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)$. Hence we get $d(\vec{v}) = (f(\vec{v}), s\langle\vec{v}\rangle \diamondsuit t_0) \mathrel{\triangleleft\!\triangleright}_i (f(\vec{v}'), s\langle\vec{v}'\rangle \diamondsuit t_0) = d(\vec{v}')$.
  - If t depends on $\mathbf{k}$, then we have $f(\vec{v}) = f(\vec{v}') =: f_0$ and $s\langle\vec{v}\rangle = s\langle\vec{v}'\rangle =: s_0$. We get $(f_0, t\langle\vec{v}\rangle) = h(\vec{v}) \mathrel{-\!\triangleright}_j h(\vec{v}') = (f_0, t\langle\vec{v}'\rangle)$. Taking a connection with s_0 yields $d(\vec{v}) = (f_0, s_0 \diamondsuit t\langle\vec{v}\rangle) \mathrel{-\!\triangleright}_j (f_0, s_0 \diamondsuit t\langle\vec{v}'\rangle) = d(\vec{v}')$. □

1.4.3 (b) Lemmas for Completeness

Proving completeness for the IPt_2 monad is fairly straightforward, but for the cases involving connections (conjunctions and disjunctions), we need a couple of helper lemmas.

Boolean reduction In this section, we establish a normal form for affine boolean terms.

Definition 1.4.3°6. Boolean **terms** $t \in \mathsf{Boo}(X)$ are equivalence classes $t = [e]$ of boolean **expressions** $e \in \mathsf{BooE}(X)$, which are defined as abstract syntax trees made up of 0, 1, $\vee$, $\wedge$, $\neg$ and elements of X. Thanks to commutativity and associativity, we regard $\vee$ and $\wedge$ as having a *multiset* of operands with at least two elements. The other operations have the usual arities.

We define a reduction algorithm that reduces an expression e to e' such that $[e] = [e']$:

- push all negations down to the leaves of the syntax tree,
- eliminate negations of constants,
- eliminate double negations,
- eliminate conjunctions/disjunctions with constants,
- remove parentheses of nested conjunctions / nested disjunctions.

We call an expression **normal** if it is its own reduction, i.e. if it is either a constant or a tree whose nodes are alternatingly (as we climb the tree) labeled with $\vee$ and $\wedge$ (the root can have either) and whose leaves are **literals**, where a literal is either a variable or its negation.

Definition 1.4.3°7. Let d and e be normal expressions. We say that d is a **pruning** of e if any of the following conditions hold (inductively):

- d is a literal occurring in e,
- e has a subexpression e' which has the same root label $\diamondsuit \in \{\vee, \wedge\}$ as d and such that there exists a partition of the multiset of operands of d, i.e.

  $$d = D_1 \diamondsuit \ldots \diamondsuit D_n \quad \text{where} \quad D_i = d_{i1} \diamondsuit \ldots \diamondsuit d_{ik_i}$$

  such that every D_i is a pruning of a root operand in e', and moreover a single operand of e' cannot be used more often than its multiplicity in the multiset of root operands of e'.

Proposition 1.4.3°8. Let e be a normal expression with variables in X and let σ be a bit-assignment $\sigma : Y \to \{0,1\}$ of the variables in $Y \subseteq X$. Let $e[\sigma]$ reduce to d. Then d is either a constant or a pruning of e.

Proof. By induction on e. If e is a constant or a literal, this is immediate. If e is a conjunction or disjunction, then this follows from the induction hypothesis for the immediate children of e. $\square$

Lemma 1.4.3°9. For every $t \in \mathsf{Boo}^{\#}(X)$ and $c \in \{0,1\}$, there exists a bit assignment $\sigma : X \to \{0,1\}$ such that $t[\sigma] = c$.

Proof. Pick an affine representant $e \in [t]$ and assume it is normal (reduce if it is not). Then we can prove this by induction on the height of e. $\square$

Lemma 1.4.3°10. For every normal affine expression $e \in \mathsf{BooE}^{\#}(X)$ mentioning all and only the variables in $Z \subseteq X$, and for every $Y \subseteq X$, there exists a bit assignment $\sigma : X \setminus Y \to \{0,1\}$ such that $e[\sigma]$ reduces to a (necessarily normal affine) expression d which mentions all and only the variables in $Y \cap Z$.

Proof. By induction on e. If e is a leaf (i.e. a constant or a literal), then this is trivial. If e is a node with label $\diamondsuit \in \{\vee, \wedge\}$, then we invoke the induction hypothesis for all immediate subtrees mentioning variables in Y, and find assignments for the variables mentioned in those immediate subtrees. For immediate subtrees not mentioning variables in Y, we use lemma 1.4.3°9 to find an assigment that reduces this subtree to the neutral element $\iota_\diamondsuit$ of $\diamondsuit$. Combining all assignments yields the required result. $\square$

Corollary 1.4.3°11. For every normal affine expression $e \in \mathsf{BooE}^{\#}(X)$ that has a leaf $\tilde{y} \in \{y, \neg y\}$ where $y \in X$, there exists a bit assignment $\sigma : X \setminus \{y\} \to \{0,1\}$ such that $t[\sigma] = \tilde{x}$. $\square$

Corollary 1.4.3°12. If e is a normal affine expression $e \in \mathsf{BooE}^{\#}(X)$ depending on y, then every expression representing $[e]$ depends on y. We say that **an affine term** $t \in \mathsf{Boo}^{\#}(X)$ **depends on** $y \in X$ if the following equivalent conditions hold:

- All representants of t depend on y.
- All normal affine representants of t depend on y.
- Some normal affine representant of t depends on y. $\square$

Definition 1.4.3°13. If e is a normal affine expression $e \in \mathsf{BooE}^{\#}(X)$ depending on $x, y \in X$, then we say that x and y are **in $\diamondsuit$-connection** (where $\diamondsuit \in \{\vee, \wedge\}$; concretely, we call this **in disjunction/conjunction**) **in** e if the closest common parent node of (the negation of) x and (the negation of) y is labelled with $\diamondsuit$. We also write this as $\mathsf{getConn}_e(x, y) = \diamondsuit$.

Lemma 1.4.3°14. Let $d, e \in \mathsf{BooE}^{\#}(X)$ be normal affine expressions and d a pruning of e. Let d (hence e) depend on $x, y \in X$. Then x and y are in $\diamondsuit$-connection in d if and only if they are in $\diamondsuit$-connection in e.

Proof. Take the closest common ancestor of x and y in d. Proving that d is a pruning of e involves, at some point, that the operands of d mentioning x and y are separated. At that point, a corresponding node in e was chosen with the same label; call e' the subtree rooted there. The immediate subtrees of d' mentioning x and y are prunings of different immediate subtrees of e', so that the root node of e' is also the closest common ancestor of x and y in e. $\square$

Lemma 1.4.3°15. An affine boolean expression $t \in \mathsf{Boo}^{\#}(X)$ mentioning exactly two variables x and y, has only a single normal representant, which is of the form $\tilde{x} \diamondsuit \tilde{y}$ with $\tilde{x}$ and $\tilde{y}$ literals mentioning x and y, and $\diamondsuit \in \{\vee, \wedge\}$.

Proof. Since every node has at least two children and there are exactly two leaves, there is exactly one node. This constrains the form of the normal representant. Next, each one of these forms produces a different truth table and the truth table is a property of t, so there can be only one normal representant. $\square$

Lemma 1.4.3°16. Let $t \in \mathsf{Boo}^{\#}(X)$ depend on $x, y \in X$, and let $\diamondsuit \in \{\vee, \wedge\}$. The following conditions are equivalent:

- x and y are in $\diamondsuit$-connection in some normal affine representant of t,
- x and y are in $\diamondsuit$-connection in all normal affine representants of t,
- x and y are **in** $\diamondsuit$**-connection in** t (definition).

We also write this as $\mathsf{getConn}_t(x, y) = \diamondsuit$.

Proof. Let e and e' be normal affine representants of t and let x and y be in $\diamondsuit$-connection in e. Pick an assignment σ such that $e[\sigma]$ reduces to d which depends exactly on $\{x, y\}$. By lemma 1.4.3°15, d is then the unique normal affine representant of $t[\sigma]$, so that $e'[\sigma]$ also reduces to d. Then d is a pruning of both e and e', so by lemma 1.4.3°14, x and y are in $\diamondsuit$-connection in e'. □

Lemma 1.4.3°17. (Not used.) Let $e \in \mathsf{BooE}^{\#}(X)$ be a normal affine expression depending on x and y where $x \neq y$. The following conditions are equivalent:

- $\mathsf{getConn}_e(x, z) = \mathsf{getConn}_e(y, z)$ for all $z \in X \setminus \{x, y\}$,
- the literals $\tilde{x}$ and $\tilde{y}$ corresponding to x and y occurring in e are immediate siblings.

Proof. It is clear that the second condition implies the first. To prove the other implication, assume that $\tilde{x}$ and $\tilde{y}$ are not immediate siblings. Let $\diamondsuit = \mathsf{getConn}_e(x, y)$. Then one of them, say $\tilde{y}$, has a closer relative $\tilde{z}$ such that $\mathsf{getConn}_e(y, z) = \heartsuit \neq \diamondsuit$. This means that e has a pruning of the form $d = \tilde{x} \diamondsuit (\tilde{y} \heartsuit \tilde{z})$. But then we have

$$\mathsf{getConn}_e(x, z) = \mathsf{getConn}_d(x, z) = \diamondsuit \neq \heartsuit = \mathsf{getConn}_d(y, z) = \mathsf{getConn}_e(y, z),$$

violating the assumption. □

Lemma 1.4.3°18. Let $X_1, X_2, \ldots, X_n \subseteq Z$ be disjoint and $n > 1$. Let $e \in \mathsf{BooE}^{\#}(Z)$ be a normal affine expression depending on all and only on variables in $X = \bigcup_i X_i$. Let $\{\diamondsuit, \heartsuit\} \in \{\vee, \wedge\}$. The following conditions are equivalent:

- both of the following conditions hold:
  - $\mathsf{getConn}_e(x, y) = \diamondsuit$ for all $x \in X_i$, $y \in X_j$, $i \neq j$,
  - each X_i is $\heartsuit$-connected meaning that no further partition of X_i is possible maintaining the property above,
- e is of the form $e = d_1 \diamondsuit d_2 \diamondsuit \ldots \diamondsuit d_n$ with each d_i depending on all and only on variables in X_i, and being either a leaf or a tree with root node labelled with $\heartsuit$.

Proof. It is clear that the second condition implies the first. To prove the other implication, assume that e is not of this form. There are two possibilities:

- e is of the form $e = d_1 \diamondsuit d_2 \diamondsuit \ldots \diamondsuit d_m$ but the dependencies are not as expected. Then we get a different partition of X by the upward implication which we already proved. But then we can intersect both partitions, yielding a finer one, which is in contradiction with the assumption that further partitioning was not possible.
- e is of the form $e = d_1 \heartsuit d_2 \heartsuit \ldots \heartsuit d_m$. Then X is $\heartsuit$-connected meaning that $n = 1$, violating the assumptions. □

Lemma 1.4.3°19. A non-constant normal affine boolean expression $e \in \mathsf{BooE}^{\#}(X)$ is fully determined by the set of literals it mentions and $\mathsf{getConn}_e(\sqcup, \sqcup)$.

Proof. Let $e, e' \in \mathsf{BooE}^{\#}(X)$ be normal affine boolean expressions mentioning the same literals and such that $\mathsf{getConn}_e(\sqcup, \sqcup) = \mathsf{getConn}_{e'}(\sqcup, \sqcup)$. We prove that $e = e'$ by induction on e. If e is a constant or a literal, the result is immediate. Otherwise, e is of the form $e = d_1 \diamondsuit d_2 \diamondsuit \ldots \diamondsuit d_n$, where $\{\diamondsuit, \heartsuit\} \in \{\vee, \wedge\}$. Then lemma 1.4.3°18 partitions X in $\heartsuit$-connected components. Then by the other implication of lemma 1.4.3°18, e' is of the form $e' = d'_1 \diamondsuit d'_2 \diamondsuit \ldots \diamondsuit d'_n$, where d_i and d'_i have the same dependencies X_i. Thus, they must also depend on the same set of literals. Since d_i is a pruning of e and d'_i is a pruning of e', we have $\mathsf{getConn}_{d_i}(\sqcup, \sqcup) = \mathsf{getConn}_e(\sqcup, \sqcup) = \mathsf{getConn}_{e'}(\sqcup, \sqcup) = \mathsf{getConn}_{d'_i}(\sqcup, \sqcup)$ when considered on pairs of distinct variables in X_i. From the induction hypothesis, we conclude that $d_i = d'_i$, for all i, and hence $e = e'$. ☐

Theorem 1.4.3°20. Every affine boolean term $t \in \mathsf{Boo}^{\#}(Z)$ has exactly one normal affine representant.

Proof. Let $e, e' \in \mathsf{BooE}^{\#}(X)$ be normal affine representants of t. By corollary 1.4.3°12, we know that e and e' depend on the same set of variables X. By corollary 1.4.3°11, we know moreover that they mention the same literals. By lemma 1.4.3°16, we know that $\mathsf{getConn}_e(\sqcup, \sqcup) = \mathsf{getConn}_{e'}(\sqcup, \sqcup) = \mathsf{getConn}_t(\sqcup, \sqcup)$. Thus, by lemma 1.4.3°19, we conclude that $e = e'$. ☐

Understanding jet cube morphisms

Lemma 1.4.3°21. In $\mathsf{JetCube}^{\square}_M(\mathsf{fbe}, \vec{a})$ with $M \in \{\mathsf{IPt}_2, \mathsf{Boo}\}$, the following holds: If a cube morphism φ is a jet cube morphism $V = (V_0, \mathbf{j} : \llparenthesis P_j \rrparenthesis, V_1) \to W = (W_0, \mathbf{i} : \llparenthesis Q_i \rrparenthesis, W_1)$ with $P, Q \in \{\rightarrowtriangle, \leftarrowtriangle, \leftrightarrowtriangle\}$ and $j > i$, and if either of the following conditions hold:

- $\mathbf{j}$ appears in $\mathbf{i}\langle\varphi\rangle$,
- $\mathbf{j}$ does not appear in φ at all,

then φ is also a jet cube morphism $\tilde{V} := (\mathsf{SymCl}^{\boxdot}_j(V_0), \mathbf{j} : \llparenthesis \leftrightarrowtriangle_j \rrparenthesis, V_1) \to W$.

We remark that this lemma is vacuous if $a_i = \bigcirc$ or $P = \leftrightarrowtriangle$.

In words, the lemma says: When a variable of the domain of a jet cube morphism is used at a lower degree in the codomain, or not at all, then that variable and all variables of the same degree to its left can be promoted to equijet variables.

Proof. Let W' be the cube obtained from W by simply deleting all variables of degree i or lower. Then the weakening morphism $\pi : W \to W'$ is a jet cube morphism. Thanks to affineness, $\pi \circ \varphi : V \to W'$ does not depend on $\mathbf{j}$. Hence, $\pi \circ \varphi \circ (0/\mathbf{j}) = \pi \circ \varphi \circ (1/\mathbf{j}) =: \rho : (\lfloor V_0 \rfloor, \lfloor V_1 \rfloor) \to \lfloor W' \rfloor$. In the category of *jet cubes* and *cube morphisms between their erasures*, we have a commutative diagram

$$\begin{tikzcd}
(\mathsf{Op}^{\boxdot}_j V_0, V_1) \arrow[r, "(0/\mathbf{j})"] \arrow[ddr, dotted, "\rho"'] & V \arrow[d, "\varphi"] & (V_0, V_1) \arrow[l, "(1/\mathbf{j})"'] \arrow[ddl, dotted, "\rho"] \\
& W \arrow[d, "\pi"] & \\
& W' &
\end{tikzcd}$$

where the black arrows are known to be jet cube morphisms, and hence the dotted arrows are also jet cube morphisms as jet cube morphisms compose. Thus, ρ is both a jet cube morphism $(\mathsf{Op}^{\boxdot}_j V_0, V_1) \to W'$ and $(V_0, V_1) \to W'$, hence it is a jet cube morphism $(\mathsf{SymCl}^{\boxdot}_j V_0, V_1) \to W'$.

We now show that φ is a jet cube morphism $\tilde{V} \to W$, i.e. that $f := \mathsf{EP}(\varphi)$ is a jet set morphism $\mathsf{JEP}(\tilde{V}) \to \mathsf{JEP}(W)$. Pick a non-reflexive jet $\vec{v} = (\vec{v}_0, u, \vec{v}_1) \rightarrowtriangle_k \vec{v}' = (\vec{v}'_0, u', \vec{v}'_1)$ in $\tilde{V}$; we show that $f(\vec{v}) \rightarrowtriangle_k f(\vec{v}')$. If $k \neq j$, then $\vec{v} \rightarrowtriangle_k \vec{v}'$ is also a jet in V and therefore preserved by f. Thus, we can assume that $k = j$. Let $\mathbf{k}$ be the unique variable where $\vec{v}$ and $\vec{v}'$ differ. There are three possibilities

$\boxed{\mathbf{k} \in V_0}$ In this case, we have $\vec{v}_0 \rightleftarrows_j \vec{v}_0'$ in V_0, $u = u'$, $\vec{v}_1 = \vec{v}_1'$, $\vec{v} \rightleftarrows_j \vec{v}'$ in V and $f(\vec{v}) \rightleftarrows_j f(\vec{v}')$ in W. If φ does not depend on $\mathbf{k}$, then $f(\vec{v}) = f(\vec{v}')$ and therefore $f(\vec{v}) \rightarrowtriangle_j f(\vec{v}')$. So we assume that φ depends on $\mathbf{k}$; let $\mathbf{l}$ be the variable such that $\mathbf{l}\langle\varphi\rangle$ depends on $\mathbf{k}$.

- If $\mathbf{l}$ is of degree $\ell \leq i < j$, then $f(\vec{v}) \rightleftarrows_j f(\vec{v}')$ is only possible if $f(\vec{v}) \rightleftarrows_\ell f(\vec{v}')$ which implies $f(\vec{v}) \leftrightarrowtriangle_j f(\vec{v}')$.
- If $\mathbf{l}$ is of degree $\ell > i$, then we have $\mathsf{EP}(\rho)(\vec{v}_0, \vec{v}_1) \overset{\neq}{\leftrightarrowtriangle}_j \mathsf{EP}(\rho)(\vec{v}_0', \vec{v}_1)$ because ρ is a jet cube morphism $(\mathsf{SymCl}_j^{\boxdot} V_0, V_1) \to W'$. Since $\mathsf{EP}(\rho) = \mathsf{EP}(\pi) \circ \mathsf{EP}(\varphi) \circ \mathsf{EP}(c/\mathbf{j})$ for any $c \in \{0, 1\}$ and the components forgotten by $\mathsf{EP}(\pi)$ are identical, we have $f(\vec{v}) \leftrightarrowtriangle_j f(\vec{v}')$.

$\boxed{\mathbf{k} = \mathbf{j}}$ In this case, we have $\vec{v}_0 = \vec{v}_0'$, $\vec{v}_1 = \vec{v}_1'$, and $\vec{v} \rightleftarrows_j \vec{v}'$ in V. Therefore we get $f(\vec{v}) \rightleftarrows_j f(\vec{v}')$ and these vectors differ at their value for $\mathbf{i}$, which has degree i, so this is only possible if $f(\vec{v}) \rightleftarrows_i f(\vec{v}')$, which implies $f(\vec{v}) \leftrightarrowtriangle_j f(\vec{v}')$.

$\boxed{\mathbf{k} \in V_1}$ Then $\vec{v} \rightarrowtriangle_j \vec{v}'$ holds in V and is therefore preserved by $f = \mathsf{EP}(\varphi)$. $\square$

Lemma 1.4.3°22. In $\mathsf{JetCube}_{\mathsf{Boo}}^{⌘}(\mathsf{fbe}, \vec{a})$ with $⌘ \in \{\square, \boxslash\}$, let V be a jet cube with only i-directed variables called (from left to right) $\mathbf{j}_1, \ldots, \mathbf{j}_n$, and consider $\varphi : V \to (\mathbf{i} : (\!|P_i|\!))$ with $P \in \{\rightarrowtriangle, \leftarrowtriangle\}$. Then $\mathbf{i}\langle\varphi\rangle$ is either a constant or of the (obviously affine) form

$$\mathbf{i}\langle\varphi\rangle = (\ldots((\neg^{p_1}\mathbf{j}_1 \diamondsuit_1 \neg^{p_2}\mathbf{j}_2) \diamondsuit_2 \neg^{p_3}\mathbf{j}_3) \ldots) \diamondsuit_{n-1} \neg^{p_n}\mathbf{j}_n$$

with $p_k \in \{0, 1\}$ and $\diamondsuit_k \in \{\vee, \wedge, \mathsf{K}\}$ where we define $x \mathbin{\mathsf{K}} y := y$.

We will only use this lemma when $⌘ = \square$.

Proof. We assume $P = \rightarrowtriangle$, the proof for $P = \leftarrowtriangle$ is analogous.

We prove this by induction on n. If $n = 0$ then $\mathbf{i}\langle\varphi\rangle$ is necessarily a constant. Assume $n > 0$, implying that $a_i = ⤢$. The jet set $\mathsf{JEP}(V)$ has 2^n elements and a unique Hamiltonian path of i-jets. The function $f := \mathsf{JEP}(\varphi)$ sends this Hamiltonian path to a path in $\mathsf{JEP}(\mathbf{i} : (\!|\rightarrowtriangle_i|\!)) = \{0 \rightarrowtriangle_i 1\}$. Thus, f is entirely determined by the step in the Hamiltonian path where the image of f flips from 0 to 1. Write $\mathbf{j}_k'$ to mean $\mathbf{j}_k$ if $\mathbf{j}_k : (\!|\rightarrowtriangle_i|\!)$ and to mean $\neg\mathbf{j}_k$ if $\mathbf{j}_k : (\!|\leftarrowtriangle_i|\!)$. There are 5 possible scenarios:

- The entire path is sent to 0. Then $\mathbf{i}\langle\varphi\rangle = 1 = \_ \mathbin{\mathsf{K}} 0$.
- The entire path is sent to 1. Then $\mathbf{i}\langle\varphi\rangle = 1 = \_ \mathbin{\mathsf{K}} 1$.
- The first half of the path is sent to 0, the second half is sent to 1. Then $\mathbf{i}\langle\varphi\rangle = \mathbf{j}_n' = \_ \mathbin{\mathsf{K}} \mathbf{j}_n'$.
- The output of f flips somewhere in the first half of the path. Then $\mathbf{i}\langle\varphi\rangle = s \vee \mathbf{j}_n'$ for some boolean expression s depending on $\mathbf{j}_1, \ldots, \mathbf{j}_{n-1}$. Write $V = (U, \mathbf{j}_n : \_)$. Then we have $(s/\mathbf{i}) : \mathsf{Op}_i^{\boxdot} U \to (\mathbf{i} : (\!|\rightarrowtriangle_i|\!))$, such that $\mathsf{JEP}(s/\mathbf{i})$ is essentially the restriction of f to the first half of the Hamiltonian path as is evident from the following commutative diagram:

$$\begin{tikzcd} \mathsf{Op}_i^{\boxdot} U \arrow[rr, "(s/\mathbf{i})"] \arrow[dr, "(0/\mathbf{j}_n)"'] & & (\mathbf{i} : (\!|\rightarrowtriangle_i|\!)). \\ & V = (U, \mathbf{j}_n : \_) \arrow[ur, "(s \vee \mathbf{j}_n/\mathbf{i})"'] & \end{tikzcd}$$

  By the induction hypothesis, s is of the prescribed form, and therefore so is $\mathbf{i}\langle\varphi\rangle$.
- The output of f flips somewhere in the second half of the path. Then $\mathbf{i}\langle\varphi\rangle = s \wedge \mathbf{j}_n'$ for some boolean expression s depending on $\mathbf{j}_1, \ldots, \mathbf{j}_{n-1}$. Write $V = (U, \mathbf{j}_n : \_)$. Then we have $(s/\mathbf{i}) : U \to (\mathbf{i} : (\!|\rightarrowtriangle_i|\!))$, such that $\mathsf{JEP}(s/\mathbf{i})$ is essentially the restriction of f to the second half of the Hamiltonian path as is evident from the following commutative diagram:

$$\begin{tikzcd} U \arrow[rr, "(s/\mathbf{i})"] \arrow[dr, "(1/\mathbf{j}_n)"'] & & (\mathbf{i} : (\!|\rightarrowtriangle_i|\!)). \\ & V = (U, \mathbf{j}_n : \_) \arrow[ur, "(s \wedge \mathbf{j}_n/\mathbf{i})"'] & \end{tikzcd}$$

By the induction hypothesis, s is of the prescribed form, and therefore so is $\mathbf{i}\langle\varphi\rangle$. □

Lemma 1.4.3°23. In $\mathsf{JetCube}^{\square}_{\mathsf{Boo}}(\mathsf{fbe}, \vec{a})$ where $a_i = \nearrow$, consider $\varphi : V \to (\mathbf{i} : (\!|P_i|\!))$ with $P \in \{\multimap, \multimapinv\}$. Write $\mathbf{j}_1, \ldots, \mathbf{j}_n$ for the i-directed variables of V. Then φ does not depend on variables of degree lower (stronger) than i, nor on i-equijet variables of V. Moreover, $\mathbf{i}\langle\varphi\rangle$ is of the form

$$\mathbf{i}\langle\varphi\rangle = h_n(h_{n-1}(\ldots h_3(h_2(h_1(\neg^{p_1}\mathbf{j}_1) \diamondsuit_1 \neg^{p_2}\mathbf{j}_2) \diamondsuit_2 \neg^{p_3}\mathbf{j}_3) \ldots) \diamondsuit_{n-1} \neg^{p_n}\mathbf{j}_n)$$

with $p_k \in \{0, 1\}$ and $\diamondsuit_k \in \{\vee, \wedge, \mathsf{K}\}$ where we define $x \mathrel{\mathsf{K}} y := y$, and every h_k is a composition of functions of the form $\sqcup \heartsuit t$ with t any affine boolean expression mentioning only i-symmetric variables and $\heartsuit \in \{\vee, \wedge, \mathsf{K}\}$.

Proof. We assume $P = \multimap$, the proof for $P = \multimapinv$ is analogous.

First of all, the i-equijet relation as well as all ℓ-jet relations for $\ell < i$ are reflexive in $\mathsf{JEP}(\mathbf{i} : (\!|\multimap_i|\!)) = \{0 \multimap_i 1\}$, so that $\mathsf{JEP}(\varphi)$ must be constant on i-equijet- or ℓ-jet-connected components, implying that φ cannot depend on those variables. Then φ factors over the map $\chi : V \to W$ that weakens over all those variables. ($\mathsf{JEP}(\chi)$ is the map that quotients out the i-equijet relation and therefore also the ℓ-jet relations for $\ell < i$.) Thus, without loss of generality, we can assume that $V = W$ contains no variables of degree lower than i, and no i-equijet variables. Moreover, applying corollary 1.4.3°2, we can assume without loss of generality that V is conventional so that the i-directed variables in V are the last ones.

Write U for the i-directed part of V and note that any assignment of bits to all i-symmetric variables yields a jet cube morphism $\sigma : U \to V$ whose composite with φ necessarily satisfies lemma 1.4.3°22. By lemma 1.4.3°10, there exists a particular assignment $\sigma_0 : U \to V$ such that $\mathbf{i}\langle\varphi \circ \sigma_0\rangle$ depends on all i-directed variables that $\mathbf{i}\langle\varphi\rangle$ depends on. The form of lemma 1.4.3°22 then dictates that $\mathbf{i}\langle\varphi\rangle$ depends on a final segment of the i-directed variables in V. Then φ factors over the map $\chi : V \to W$ that weakens over the initial segment of i-directed variables that φ does not depend on, and again, without loss of generality, we assume that $V = W$, i.e. that φ uses *all* i-directed variables in V.

The fact that $\mathbf{i}\langle\varphi \circ \sigma_0\rangle$ satisfies lemma 1.4.3°22 and is a pruning of $\mathbf{i}\langle\varphi\rangle$ (by proposition 1.4.3°8), severely constrains the possible forms that $\mathbf{i}\langle\varphi\rangle$ may take. However, lemma 1.4.3°22 does not require that conjunction and disjunction appear in alternation, and thus, a priori, by associativity, parentheses could be moved around before unpruning. Thus, we need to constrain further to obtain the form in the current lemma.

Let e be the unique normal affine representant of $\mathbf{i}\langle\varphi\rangle$ (theorem 1.4.3°20). We proceed by induction on e. If e is a constant or i-symmetric literal, then it is of the required form. If it is an i-directed literal, then it is the only one it depends on, hence $\neg^{p_n}\mathbf{j}'_n$ with $n = 1$, and thus of the required form.

If e is of the form $e = d_1 \diamondsuit \ldots \diamondsuit d_m$, then we need to prove that either $\neg^{p_n}\mathbf{j}_n$ is a direct operand of e, or all i-directed variables occur in the same direct operand of e. Suppose that neither is the case. Then we can assume that d_1 depends *indirectly* on $\mathbf{j}_n$ and d_2 depends on $\mathbf{j}_k$. Say $d_1 = c_1 \clubsuit \ldots \clubsuit c_\ell$ with $\{\diamondsuit, \clubsuit\} = \{\vee, \wedge\}$ and c_1 depends on $\mathbf{j}_n$. There is an assignment σ_1 of the i-symmetric dependencies of d_1 that reduces all $c_{i>2}$ to the absorbing element $\infty_\clubsuit$ of $\clubsuit$, and thus reduces d_1 to $\infty_\clubsuit = \iota_\diamondsuit$, the neutral element of $\diamondsuit$. For $i > 2$, there is an assignment σ_i of the i-symmetric dependencies of d_i that preserves all i-directed dependencies. We combine all these assignments to a single assignment $\sigma : U \to V$. Then $e[\sigma]$ reduces to a normal expression that depends on $\mathbf{j}_k$ but not $\mathbf{j}_n$, which is then a representant of $\mathbf{i}\langle\varphi \circ \sigma\rangle$, which we know must satisfy lemma 1.4.3°22 and thus cannot depend on $\mathbf{j}_k$ without also depending on $\mathbf{j}_n$. □

Lemma 1.4.3°24. In $\mathsf{JetCubeConv}^{\square}_{\mathsf{Boo}}(\mathsf{fbe}, \vec{a})$, let $\hat{\varphi} : V \to W$ be a jet cube morphism and write $\varphi = \lfloor\hat{\varphi}\rfloor$. Let $W = (W_0, \mathbf{i} : (\!|Q_i|\!))$ with $Q \in \{\multimap, \multimapinv\}$ and $a_i = \nearrow$. Let e be the normal affine representant of $\mathbf{i}\langle\varphi\rangle$ and let $\mathbf{j}_1, \ldots, \mathbf{j}_n$ ($n \geq 0$) be all the variables of degree i that e depends on, and $\mathbf{k}_1, \ldots, \mathbf{k}_m$ ($m \geq 0$) all the other variables that e depends on. Assume $m + n \geq 2$, so that e necessarily contains a conjunction or disjunction. By lemma 1.4.3°23, we know that $\mathbf{j}_1, \ldots, \mathbf{j}_n$ are the *last* n variables of degree i in V. Write $V = (V_0, \mathbf{j}_1 : (\!|P^1_i|\!), \ldots, \mathbf{j}_n : (\!|P^n_i|\!), V_1)$ so that (even if $n = 0$) all variables in V_0 have degree at least (at the strongest) i and all variables in V_1 have degree strictly less (stronger) than i. Here, each $P^1, \ldots, P^n \in \{\multimap, \multimapinv\}$.[8] Define $\tilde{V} = (\mathsf{SymCl}^{\boxdot}_i V_0, \mathbf{j}_1 : (\!|P^1_i|\!), \ldots, \mathbf{j}_n : (\!|P^n_i|\!), V_n)$, i.e. every variable of degree i to the left of $\mathbf{j}_1$ gets promoted to an equijet variable. Then φ is a jet cube morphism $\tilde{V} \to W$.

[8] We could in principle allow $\leftrightarrow$ but it is easy to see that φ being a jet cube morphism (equivalently, $\mathsf{EP}(\varphi)$ being a jet set) implies $P^1, \ldots, P^n \in \{\multimap, \multimapinv\}$.

It is clear that $\mathbf{k}_1, \dots, \mathbf{k}_m$ all occur in V_0 as they cannot have degree less (stronger) than i.

In words, this lemma says that if the last variable $\mathbf{i}$ of W is substituted with an expression e depending on at least two variables, then all variables in V of same degree as $\mathbf{i}$ that e does *not* depend on, can be promoted to equijet variables.

Proof. For $c \in \{0,1\}$, let A_c be the set of all $(\vec{\kappa}, \vec{\zeta}) \in \{0,1\}^{m+n}$ such that $\mathbf{i}\langle\varphi\rangle\langle\vec{\kappa}/\vec{\mathbf{k}}, \vec{\zeta}/\vec{\mathbf{j}}\rangle = c$. Let U be the (ordinary) cube obtained from $\lfloor V \rfloor$ by removing all dependencies of e. Then for any $(\vec{\kappa}, \vec{\zeta}) \in \{0,1\}^{m+n}$, by applying cube opposite functors in all the right places, there is a jet cube $U_{(\vec{\kappa},\vec{\zeta})}$ such that $\left\lfloor U_{(\vec{\kappa},\vec{\zeta})} \right\rfloor = U$ and $(\vec{\kappa}/\vec{\mathbf{k}}, \vec{\zeta}/\vec{\mathbf{j}}) : U_{(\vec{\kappa},\vec{\zeta})} \to V$ is a jet cube morphism.

Then in the category of *jet cubes* and *cube morphisms between their erasures*, for any $(\vec{\kappa}, \vec{\zeta}) \in A_c$, we obtain a commutative diagram

$$\begin{tikzcd}
U_{(\vec{\kappa},\vec{\zeta})} \arrow[r, dotted, "\chi"] \arrow[d, "(\vec{\kappa}/\vec{\mathbf{k}},\vec{\zeta}/\vec{\mathbf{j}})"'] & (\mathsf{Op}_i^{\boxdot})^{1-c}(W_0) \arrow[d, "(c/\mathbf{i})"] \\
V \arrow[r, "\varphi"'] & W
\end{tikzcd}$$

where all the black lines are jet cube morphisms and the cube morphism χ is defined as $(\mathbf{i}/\oslash) \circ \varphi \circ (\vec{\kappa}/\vec{\mathbf{k}}, \vec{\zeta}/\vec{\mathbf{j}})$, which thanks to afineness does not depend on our choice of $(\vec{\kappa}, \vec{\zeta})$, nor even on c. Commutativity of the diagram and the fact that $\mathsf{JEP}(c/\mathbf{i})$ is a full jet set morphism, imply that χ, too, is a jet cube morphism.

We now show that φ is a jet cube morphsim $\tilde{V} \to W$. Let $\vec{v} \rightarrowtriangle_k \vec{v}'$ in $\tilde{V}$. We prove that $f(\vec{v}) \rightarrowtriangle_k f(\vec{v}')$ where $f = \mathsf{EP}(\varphi)$. If $\vec{v} = \vec{v}'$ then this is trivial, so let $\mathbf{l}$ be the variable where they differ. If $k \neq i$ or $\mathbf{l} \notin V_0$ then we have $\vec{v} \rightarrowtriangle_k \vec{v}'$ in V so this is preserved by f.

Assume $k = i$ and $\mathbf{l} \in V_0$, which implies that $\mathbf{l}$ is not a dependency of e. This implies that $e\langle\vec{v}\rangle = e\langle\vec{v}'\rangle =: c$, or differently put $\mathbf{i}\langle f(\vec{v})\rangle = \mathbf{i}\langle f(\vec{v}')\rangle = c$. We have $\vec{v} \rightleftarrows_i \vec{v}'$ in V, say $\vec{v}\ S_i\ \vec{v}'$ where $S \in \{\rightarrowtriangle, \leftarrowtriangle\}$. This is preserved by f, so $f(\vec{v})\ S_i\ f(\vec{v}')$. Because $(c/\mathbf{i})$ is a full jet set morphism, writing $p = \mathsf{EP}(\mathbf{i}/\oslash)$, we can conclude that $p(f(\vec{v}))\ S_i\ p(f(\vec{v}'))$ in $(\mathsf{Op}_i^{\boxdot})^{1-c}(W_0)$.

Let $\vec{u}$ and $\vec{u}'$ be bit-assignments to the variables in U obtained by projecting out all bits assigned to the dependencies of e in the vectors $\vec{v}$ and $\vec{v}'$, and let $\vec{\zeta}$ and $\vec{\kappa}$ be the bits thus forgotten (which are the same for $\vec{v}$ and $\vec{v}'$). Thus, $\vec{v} = \mathsf{EP}(\vec{\zeta}/\vec{\mathbf{j}}, \vec{\kappa}/\vec{\mathbf{k}})(\vec{u})$ and similar for $\vec{v}'$. Writing $g = \mathsf{EP}(\chi)$, this implies that $p(f(\vec{v})) = g(\vec{u})$ and $p(f(\vec{v}')) = g(\vec{u}')$. Thus, we have $g(\vec{u})\ S_i\ g(\vec{u}')$ in $(\mathsf{Op}_i^{\boxdot})^{1-c}(W_0)$.

If φ does not depend on $\mathbf{l}$, then we have nothing to prove, so let $\mathbf{h}$ be the variable of W such that $\mathbf{h}\langle\varphi\rangle$ depends on $\mathbf{l}$. Note that $\mathbf{h} \neq \mathbf{i}$. We remark that the direction of the jet between $g(\vec{u})$ and $g(\vec{u}')$ in $(\mathsf{Op}_i^{\boxdot})^{1-c}(W_0)$ flips with $c = e\langle\vec{v}\rangle$, which is a function of $(\vec{\kappa}, \vec{\zeta})$.

On the other hand, looking at the direction of the jet between $\vec{u}$ and $\vec{u}'$ in $U_{(\vec{\kappa},\vec{\zeta})}$, we see that this flips with the $\mathbf{j}_1 \veebar \dots \veebar \mathbf{j}_n$, the exclusive disjunction of all dependencies of e of degree i, which all appear to the right of $\mathbf{l}$. Now if $m + n \geq 2$, then it is impossible that the affine boolean expression $e \in \mathsf{Boo}^{\#}(\{\mathbf{k}_1, \dots, \mathbf{k}_m, \mathbf{j}_1, \dots, \mathbf{j}_n\})$ which is in a reduced state and therefore truly depends on each of the mentioned variables, yields the exact same (or opposite) truth table as $\mathbf{j}_1 \veebar \dots \veebar \mathbf{j}_n$.

Thus, fixing $\vec{u}$ and $\vec{u}'$ and varying $(\vec{\kappa}, \vec{\zeta})$, we see that there are assignments $(\vec{\kappa}, \vec{\zeta})$ for which the jets between $\vec{u}$ and $\vec{u}'$ in $U_{(\vec{\kappa},\vec{\zeta})}$ on one hand, and between $g(\vec{u})$ and $g(\vec{u}')$ in $(\mathsf{Op}_i^{\boxdot})^{1-c}(W_0)$ are aligned, and others for which they are opposed. Pick an assignment $(\vec{\kappa}', \vec{\zeta}')$ for which they are opposed. We also know that χ is a jet cube morphism for any assignment $(\vec{\kappa}, \vec{\zeta})$, and in particular for $(\vec{\kappa}', \vec{\zeta}')$. Thus, χ provides us the jet pointing the other way, and we can conclude that $g(\vec{u}) \leftrightarrowtriangle_i g(\vec{u}')$. Composing with $\mathsf{EP}(c/\mathbf{i})$ for our original c yields $f(\vec{v}) \leftrightarrowtriangle_i f(\vec{v}')$. $\square$

Corollary 1.4.3°25. In $\mathsf{JetCube}^{\square}_{\mathsf{Boo}}(\mathsf{fbe}, \vec{a})$, let $\hat{\varphi} : V \to W$ be a jet cube morphism and write $\varphi = \lfloor \hat{\varphi} \rfloor$. Let $W = (W_0, \mathbf{i} : \llparenthesis Q_i \rrparenthesis)$ with $Q \in \{\rightarrowtriangle, \leftarrowtriangle\}$ and $a_i = \nearrow$. Let e be the reduction of $\mathbf{i}\langle\varphi\rangle$ and assume e depends on at least two variables. Then $(\oslash/\mathbf{i}) \circ \varphi$ is a jet cube morphism $\mathsf{SymCl}_i^{\boxdot} V \to W_0$.

Proof. We know from lemma 1.4.3°24 that φ is a jet cube morphism $\tilde{V} \to W$. We first show that $(\oslash/\mathbf{i}) \circ \varphi$ is a jet cube morphism $\tilde{V} \to W_0$. Write $f = \mathsf{EP}(\varphi)$ and $p = \mathsf{EP}(\oslash/\mathbf{i})$. Pick a non-reflexive jet $\vec{v} \rightarrowtriangle_j \vec{v}'$ in $\tilde{V}$. We show that $p(f(\vec{v})) \rightarrowtriangle_j p(f(\vec{v}'))$ in W_0. We know that $f(\vec{v}) \rightarrowtriangle_j f(\vec{v}')$ in W. If $j \neq i$, it follows that $p(f(\vec{v})) \rightarrowtriangle_j p(f(\vec{v}'))$ in W_0. If $j = i$, it follows that $p(f(\vec{v})) \rightleftarrows_i p(f(\vec{v}'))$ in W_0. But because $(\oslash/\mathbf{i}) \circ \varphi$ only depends on i-symmetric variables in $\tilde{V}$, we can conclude that $p(f(\vec{v})) \leftrightarrowtriangle_i p(f(\vec{v}'))$.

We conclude that $(\oslash/\mathbf{i}) \circ \varphi$ is a jet cube morphism $\tilde{V} \to W$. Since it only depends on i-symmetric variables, there is no harm in promoting the ignored variables of degree i to equijet variables, and that is all that $\mathsf{SymCl}_i^{\boxdot} V$ does. $\square$

1.4.3 (c) Completeness

Theorem 1.4.3°26 (Completeness)**.** For $M \in \{\mathsf{IPt}_2, \mathsf{Boo}\}$ and any morphism $\hat{\varphi} : V \to W$ in $\mathsf{JetCubeConv}_M^{\boxdot}(\mathsf{fbe}, \vec{a})$, writing $\varphi = \lfloor \hat{\varphi} \rfloor$, we have $\vdash \varphi : V \to W$.

Proof. For each variable $\mathbf{k}$ in W, let $e_{\mathbf{k}}$ be the normal affine representant of $\mathbf{k}\langle\varphi\rangle$.

We prove completeness by induction on the number of nodes and leaves (added up) in the tuple $(e_{\mathbf{k}})_{\mathbf{k} \in W}$, plus the number of directed degrees in the mask.

If $W = ()$, then use TERMINAL.

If $a_i = \nearrow$ and the last variable in W is $\mathbf{i} : \llparenthesis \leftrightarrowtriangle_i \rrparenthesis$, then W is of the form $\mathsf{USym}_i^{\boxdot}(U)$ and we can use the induction hypothesis for the corresponding morphism $\mathsf{FSym}_i^{\boxdot}(V) \to U$ and we can use the rule SYMMETRIZE.

In the remaining case, the last variable in W is not an equijet dimension at a directed degree, i.e. it is of the form $\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis$ or $\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis$.

If the last variable in V is of degree strictly lower (stronger) than the degree of $\mathbf{i}$, then in order to be a jet set morphism, $\mathsf{JEP}(\hat{\varphi})$ cannot depend on that variable, so we can invoke WKN until the last variable in V is of degree at least i. We do not resort to the induction hypothesis but proceed below.

We proceed by inspecting $e_{\mathbf{i}}$.

- If $[\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis$ and $e_{\mathbf{i}} = 0]$ or $[\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis$ and $e_{\mathbf{i}} = 1]$, then φ being a jet cube morphism $V \to W = (U, \mathbf{i} : \_)$ (equivalently: $\mathsf{EP}(\varphi)$ being a jet set morphism $\mathsf{JEP}(V) \to \mathsf{JEP}(W)$) is equivalent to $(\oslash/\mathbf{i}) \circ \varphi$ being a jet cube morphism $V \to \mathsf{Op}_i^{\boxdot}(U)$, so we can apply SRC:FWD or SRC:BCK.
- If $[\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis$ and $e_{\mathbf{i}} = 1]$ or $[\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis$ and $e_{\mathbf{i}} = 0]$, then φ being a jet cube morphism $V \to W = (U, \mathbf{i} : \_)$ (equivalently: $\mathsf{EP}(\varphi)$ being a jet set morphism $\mathsf{JEP}(V) \to \mathsf{JEP}(W)$) is equivalent to $(\oslash/\mathbf{i}) \circ \varphi$ being a jet cube morphism $V \to U$, so we can apply TGT:FWD or TGT:BCK.
- If $\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis$ and $e_{\mathbf{i}} = \neg\mathbf{j}$, then φ being a jet cube morphism $V \to W = (U, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$ is equivalent to $(\neg\mathbf{i}/\mathbf{i}) \circ \varphi$ being a jet cube morphism $V \to (U, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis)$, so we can apply INV:FWD. Similarly, if $\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis$ and $e_{\mathbf{i}} = \neg\mathbf{j}$, we can apply INV:BCK.
- If $\mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis$ ($\mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis$ is handled analogously) and $e_{\mathbf{i}} = \mathbf{j}$ where V specifies that $\mathbf{j}$ has degree j, then we know that $\varphi = (\chi, \mathbf{j}/\mathbf{i})$ is a jet cube morphism $V = (V_0, \mathbf{j} : \llparenthesis R_j \rrparenthesis, V_1) \to W = (U, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$ for some $R \in \{\rightarrowtriangle, \leftarrowtriangle, \leftrightarrowtriangle\}$. We have the following commutative diagram in the category of *jet cubes* and *cube morphisms between erased jet cubes*:

$$
\begin{tikzcd}
(\mathsf{Op}_j^{\boxdot}(V_0), V_1) \arrow[r, dotted, "\chi"] \arrow[d, "(0/\mathbf{j})"'] & \mathsf{Op}_i^{\boxdot}(U) \arrow[d, "(0/\mathbf{i})"] \\
(V_0, \mathbf{j} : \llparenthesis R_j \rrparenthesis, V_1) \arrow[r, "\varphi = (\chi, \mathbf{j}/\mathbf{i})"'] & (U, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \\
(V_0, V_1) \arrow[r, dotted, "\chi"'] \arrow[u, "(1/\mathbf{j})"] & U \arrow[u, "(1/\mathbf{i})"']
\end{tikzcd}
$$

  We know that the black arrows are all jet cube morphisms, and the vertical arrows all yield full jet set morphisms (definition 1.2.1°2). This implies that the dashed arrows also lift to jet set morphisms, i.e. are jet cube morphisms. Then χ is both a jet cube morphism $(V_0, V_1) \to U$ and $\mathsf{Op}_i^{\boxdot}(\mathsf{Op}_j^{\boxdot}(V_0), V_1) \to U$.

– If $j = i$, then all variables in V_1 have degree i and χ is both a jet cube morphism $(V_0, V_1) \to U$ and $\mathsf{Op}_i^{\boxdot}(\mathsf{Op}_i^{\boxdot} V_0, V_1) \to U$. This implies that $\mathsf{EP}(\varphi)$ sends every i-jet of the form $(\vec{v}_0, r, \vec{v}_1) \mathrel{-\!\!\vartriangleright}_i (\vec{v}_0, r, \vec{v}_1')$ in $\mathsf{JEP}(V_0, V_1)$ – which points the other way in $\mathsf{JEP}(\mathsf{Op}_i^{\boxdot}(\mathsf{Op}_i^{\boxdot} V_0, V_1))$ – to an i-equijet $(\mathsf{EP}(\chi)(\vec{v}_0, \vec{v}_1), r) \mathrel{\vartriangleleft\!\!\vartriangleright}_i (\mathsf{EP}(\chi)(\vec{v}_0, \vec{v}_1'), r)$ in $\mathsf{JEP}(U, \mathbf{i} : (\!| \mathrel{-\!\!\vartriangleright}_i |\!))$.

  * If $a_i = \nearrow$, then for any bit-assignment $\vec{v}_1$ of the variables in V_1, we get a jet cube morphism $(\chi \circ (\vec{v}_1/V_1), \mathbf{j}/\mathbf{i}) : (\mathsf{Op}_i^{\boxdot})^p(V_0, \mathbf{j} : (\!| R_i |\!)) \to (U, \mathbf{i} : (\!| \mathrel{-\!\!\vartriangleright}_i |\!))$, where p is the number of source constants in $\vec{v}$ (i.e. 0 for $(\!| \mathrel{-\!\!\vartriangleright}_i |\!)$ and 1 for $(\!| \mathrel{\vartriangleleft\!\!-}_i |\!)$). This implies that p is the same for all assignments $\vec{v}$, which is only possible if $V_1 = ()$. In that case, it is easy to see that $R = \mathrel{-\!\!\vartriangleright}$. Thus, we can apply PRISM:FWD.
  * If $a_i = \bigcirc$, then we can use EXCHANGE to create a morphism from $(V_0, V_1, \mathbf{j} : (\!| \frown_i |\!))$ instead, which can be done using PRISM:FWD (or equivalently PRISM:BCK).

  If $j > i$, then χ is necessarily a jet cube morphism $\mathsf{SymCl}_i^{\boxdot}(\mathsf{SymCl}_j^{\boxdot}(V_0), V_1) \to U$, so we can apply CONCURSOR.

- We have now covered all cases for the monad IPt_2. In the remaining cases, $e_\mathbf{i}$ contains connection (conjunction or disjunction) symbols. If $\mathbf{i} : (\!| \mathrel{\vartriangleleft\!\!-}_i |\!)$, we apply INV:FWD and push down the introduced negation, after which we do not resort to the induction hypothesis but proceed below.[9] We now assume that $\mathbf{i} : (\!| \mathrel{-\!\!\vartriangleright}_i |\!)$.

  – We first treat the case where $a_i = \bigcirc$. Let $\varphi = (\chi, s \Diamond t/\mathbf{i})$ where $\Diamond \in \{\vee, \wedge\}$ and $W = (U, \mathbf{i} : (\!| \mathrel{-\!\!\vartriangleright}_i |\!))$. We claim that if φ is a jet cube morphism $V \to W = (U, \mathbf{i} : (\!| \frown_i |\!))$, then so are $(\chi, s/\mathbf{i})$ and $(\chi, t/\mathbf{i})$, so that we can invoke CONN:SYM. Write

  $$f = \mathsf{EP}(\varphi), \qquad g = \mathsf{EP}(\chi), \qquad p = \mathsf{EP}(\chi, s/\mathbf{i}), \qquad q = \mathsf{EP}(\chi, t/\mathbf{i}).$$

  Pick a non-reflexive jet $\vec{v} \mathrel{-\!\!\vartriangleright}_j \vec{v}'$. We prove $p(\vec{v}) \mathrel{-\!\!\vartriangleright}_j p(\vec{v}')$; by symmetry of the situation we do not have to prove the same for q. Let $\mathbf{k}$ be the variable of V where $\vec{v}$ and $\vec{v}'$ differ. There are four possible situations:

  * If φ does not depend on $\mathbf{k}$, then we are done.
  * If χ depends on $\mathbf{k}$, then we have $s\langle\vec{v}\rangle = s\langle\vec{v}'\rangle =: s_0$ and $t\langle\vec{v}\rangle = t\langle\vec{v}'\rangle =: t_0$.
    · If $j \neq i$, then from $(g(\vec{v}), s_0 \Diamond t_0) = f(\vec{v}) \mathrel{-\!\!\vartriangleright}_j f(\vec{v}') = (g(\vec{v}'), s_0 \Diamond t_0)$, it follows that $g(\vec{v}) \mathrel{-\!\!\vartriangleright}_j g(\vec{v}')$ in $\mathsf{JEP}(U)$, whence $p(\vec{v}) = (g(\vec{v}), s_0) \mathrel{-\!\!\vartriangleright}_j (g(\vec{v}'), s_0) = p(\vec{v}')$.
    · If $j = i$, then from $(g(\vec{v}), s_0 \Diamond t_0) = f(\vec{v}) \frown_i f(\vec{v}') = (g(\vec{v}'), s_0 \Diamond t_0)$ it follows that $g(\vec{v}) \frown_i g(\vec{v}')$ in $\mathsf{JEP}(U)$, whence $p(\vec{v}) = (g(\vec{v}), s_0) \frown_i (g(\vec{v}'), s_0) = p(\vec{v}')$.
  * If s depends on $\mathbf{k}$, then we have $g(\vec{v}) = g(\vec{v}') =: g_0$ and $t\langle\vec{v}\rangle = t\langle\vec{v}'\rangle =: t_0$. Pick a bit-assignment τ of the dependencies of t such that $t\langle\tau\rangle$ reduces to the neutral element $\iota_\Diamond$ of $\Diamond$.[10] Define $\vec{x}$ and $\vec{x}'$ by overwriting $\vec{v}$ and $\vec{v}'$ with τ. Then $g(\vec{x}) = g(\vec{x}') = g_0$ and $t\langle\vec{x}\rangle = t\langle\vec{x}'\rangle = \iota_\Diamond$ and $s\langle\vec{x}\rangle = s\langle\vec{v}\rangle$ and $s\langle\vec{x}'\rangle = s\langle\vec{v}'\rangle$. We have $\vec{x} \rightleftarrows_j \vec{x}'$, whence

  $$\begin{array}{ccccccccc} p(\vec{v}) & = & (g_0, s\langle\vec{v}\rangle) & = & (g_0, s\langle\vec{v}\rangle \Diamond \iota_\Diamond) & = & (g_0, s\langle\vec{x}\rangle \Diamond t\langle\vec{x}\rangle) & = & f(\vec{x}) \\ & & & & & & & & \rightleftarrows_j \\ p(\vec{v}') & = & (g_0, s\langle\vec{v}'\rangle) & = & (g_0, s\langle\vec{v}'\rangle \Diamond \iota_\Diamond) & = & (g_0, s\langle\vec{x}'\rangle \Diamond t\langle\vec{x}'\rangle) & = & f(\vec{x}') \end{array}$$

  in $\mathsf{JEP}(W, \mathbf{i} : (\!| \mathrel{-\!\!\vartriangleright}_i |\!))$. Now, since these vectors only differ at $\mathbf{i}$, we can deduce equality if $j < i$ (j is stronger than i) and otherwise that $p(\vec{v}) \frown_i p(\vec{v}')$, which implies $p(\vec{v}) \mathrel{-\!\!\vartriangleright}_j p(\vec{v}')$ since $j \geq i$ (j is weaker than or equal to i).
  * If t depends on $\mathbf{k}$, then χ and s do not so $p(\vec{v}) = p(\vec{v}')$ and we are done.

  – Now assume that $a_i = \nearrow$. Let $\varphi = (\chi, t \Diamond s/\mathbf{i})$ where $\Diamond \in \{\vee, \wedge\}$ and $W = (U, \mathbf{i} : (\!| \mathrel{-\!\!\vartriangleright}_i |\!))$. Corollary 1.4.3°25 immediately tells us that χ is a jet cube morphism $\mathsf{SymCl}^{\boxdot} V \to U$. By lemma 1.4.3°23, we can assume that s is either (the negation of) the last variable of degree i in $V = (V_0, \mathbf{i} : (\!| P_i |\!))$ with $P \in \{\mathrel{-\!\!\vartriangleright}, \mathrel{\vartriangleleft\!\!-}\}$ or a boolean expression only depending on i-symmetric variables.

[9] Alternatively, we could duplicate and adapt the proof below to the case where $\mathbf{i} : (\!| \mathrel{\vartriangleleft\!\!-}_i |\!)$.

[10] If this were not possible, then t would be a constant, which is in contradiction with the assumption that $e_\mathbf{i}$ was normal.

- If $s = \mathbf{i}' \in \{\mathbf{i}, \neg\mathbf{i}\}$, then depending on $\diamondsuit$ we have one of the following commutative diagrams, where each arrow is a jet cube morphism but labelled with its erasure:

$$\begin{tikzcd}
 & \mathsf{Op}^{\boxdot}_i V \arrow[d, "(0/\mathbf{i})"'] \arrow[dr, bend left=10, "(\chi{,}t/\mathbf{i})"] & \\
(V, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis) \arrow[r, leftrightarrow, "\cong", "(\chi{,}\neg\mathbf{i}/\mathbf{i})"'] & (V, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \arrow[r, "(\chi{,}t\vee\mathbf{i}/\mathbf{i})"'] & (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)
\end{tikzcd}$$

$$\begin{tikzcd}
 & V \arrow[d, "(1/\mathbf{i})"'] \arrow[dr, bend left=10, "(\chi{,}t/\mathbf{i})"] & \\
(V, \mathbf{i} : \llparenthesis \leftarrowtriangle_i \rrparenthesis) \arrow[r, leftrightarrow, "\cong", "(\chi{,}\neg\mathbf{i}/\mathbf{i})"'] & (V, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis) \arrow[r, "(\chi{,}t\wedge\mathbf{i}/\mathbf{i})"'] & (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)
\end{tikzcd}$$

  Thus, we can invoke one of the rules CONN:PRISM:SRC-NEUTRAL, CONN:PRISM:TGT-NEUTRAL, CONN:PRISM-INV:SRC-NEUTRAL, CONN:PRISM-INV:TGT-NEUTRAL.
- If s depends only on i-symmetric variables, then the same holds for $(\chi, s/\mathbf{i})$. Write

$$f = \mathsf{EP}(\varphi), \qquad g = \mathsf{EP}(\chi), \qquad p = \mathsf{EP}(\chi, s/\mathbf{i}), \qquad q = \mathsf{EP}(\chi, t/\mathbf{i}).$$

  We show that
  - $(\chi, s/\mathbf{i})$ is a jet cube morphism $\mathsf{SymCl}^{\boxdot}_i V \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$,
  - $(\chi, t/\mathbf{i})$ is a jet cube morphism $V \to (W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$,

  so that we can invoke CONN:DEGREE-SYMMETRIC.
  Pick a non-reflexive jet $\vec{v} \rightarrowtriangle_j \vec{v}'$ in $\mathsf{JEP}(V)$. We will prove $p(\vec{v}) \rightarrowtriangle_j p(\vec{v}')$ and $q(\vec{v}) \rightarrowtriangle_j q(\vec{v}')$ and, if $j = i$, even $p(\vec{v}) \leftrightarrowtriangle_j p(\vec{v}')$, all the time in $\mathsf{JEP}(W, \mathbf{i} : \llparenthesis \rightarrowtriangle_i \rrparenthesis)$. Let $\mathbf{k}$ be the variable where $\vec{v}$ and $\vec{v}'$ differ. There are four possibilities:
  - If φ does not depend on $\mathbf{k}$, then we are done.
  - If χ depends on $\mathbf{k}$, then $s\langle\vec{v}\rangle = s\langle\vec{v}'\rangle =: s_0$ and $t\langle\vec{v}\rangle = t\langle\vec{v}'\rangle =: t_0$.
    - If $j = i$, then $g(\vec{v}) \leftrightarrowtriangle_i g(\vec{v}')$ and hence $p(\vec{v}) = (g(\vec{v}), s_0) \leftrightarrowtriangle_i (g(\vec{v}'), s_0) = p(\vec{v}')$ and $q(\vec{v}) = (g(\vec{v}), t_0) \leftrightarrowtriangle_i (g(\vec{v}'), t_0) = q(\vec{v}')$ as required.
    - If $j \neq i$, then $g(\vec{v}) \rightarrowtriangle_j g(\vec{v}')$ and hence $p(\vec{v}) = (g(\vec{v}), s_0) \rightarrowtriangle_j (g(\vec{v}'), s_0) = p(\vec{v}')$ and $q(\vec{v}) = (g(\vec{v}), t_0) \rightarrowtriangle_j (g(\vec{v}'), t_0) = q(\vec{v}')$ as required.
  - If s depends on $\mathbf{k}$, then $g(\vec{v}) = g(\vec{v}') =: g_0$ and $t\langle\vec{v}\rangle = t\langle\vec{v}'\rangle =: t_0$, so $q(\vec{v}) = q(\vec{v}')$. We pick τ and define $\vec{x}$ and $\vec{x}'$ in the same way as we did when $a_i = \bigcirc$ and all other circumstances were the same. Then we have $\vec{x} \rightleftarrows_j \vec{x}'$ in $\mathsf{JEP}(V)$, whence $p(\vec{v}) = f(\vec{x}) \rightleftarrows_j f(\vec{x}') = p(\vec{v}')$. Now, since these vectors only differ at $\mathbf{i}$, we can deduce equality if $j < i$ (j is stronger than i) and otherwise that $p(\vec{v}) \rightleftarrows_i p(\vec{v}')$. But since s and χ only depend on i-symmetric variables, it must be the case that $p(\vec{v}) \leftrightarrowtriangle_i p(\vec{v}')$, as required if $i = j$, and implying $p(\vec{v}) \rightarrowtriangle_j p(\vec{v}')$ if $j > i$.
  - If t depends on $\mathbf{k}$, then $g(\vec{v}) = g(\vec{v}') =: g_0$ and $s\langle\vec{v}\rangle = s\langle\vec{v}'\rangle =: s_0$, so $p(\vec{v}) = p(\vec{v}')$. Pick an assignment σ of the dependencies of s such that $s\langle\sigma\rangle$ reduces to the neutral element $\iota_\diamondsuit$ of $\diamondsuit$. Define $\vec{y}$ and $\vec{y}'$ by overwriting $\vec{v}$ and $\vec{v}'$ with σ. Then $g(\vec{y}) = g(\vec{y}') = g_0$ and $s\langle\vec{y}\rangle = s\langle\vec{y}'\rangle = \iota_\diamondsuit$ and $t\langle\vec{y}\rangle = t\langle\vec{v}\rangle$ and $t\langle\vec{y}'\rangle = t\langle\vec{v}'\rangle$. We have $\vec{y} \rightleftarrows_j \vec{y}'$ but, since all dependencies of s are i-symmetric, they come to the left of all i-directed variables, so if $i = j$ we actually still have $\vec{y} \rightarrowtriangle_i \vec{y}'$. Now we have

$$\begin{array}{ccccccccc}
q(\vec{v}) & = & (g_0, t\langle\vec{v}\rangle) & = & (g_0, \iota_\diamondsuit \diamondsuit t\langle\vec{v}\rangle) & = & (g_0, s\langle\vec{y}\rangle \diamondsuit t\langle\vec{y}\rangle) & = & f(\vec{y}) \\
 & & & & & & & & \rightleftarrows_j \\
q(\vec{v}') & = & (g_0, t\langle\vec{v}'\rangle) & = & (g_0, \iota_\diamondsuit \diamondsuit t\langle\vec{v}'\rangle) & = & (g_0, s\langle\vec{y}'\rangle \diamondsuit t\langle\vec{y}'\rangle) & = & f(\vec{y}')
\end{array}$$

    and $q(\vec{v}) \rightarrowtriangle_i q(\vec{v}')$ if $j = i$. Thus, the case $j = i$ has been handled. Since $q(\vec{v})$ and $q(\vec{v}')$ can only differ at $\mathbf{i}$ which has degree i, we can deduce equality if $j < i$ (j is stronger than i) and otherwise that $q(\vec{v}) \rightleftarrows_i q(\vec{v}')$ which implies $q(\vec{v}) \rightarrowtriangle_j q(\vec{v}')$ as required. □

1.4.4 The Semisymmetric Separated Product

Definition 1.4.4°1. We define the **separated product** functor

$$\sqcup * \sqcup : \mathsf{JetSet}(\vec{a}) \times \mathsf{JetSet}(\vec{a}) \to \mathsf{JetSet}(\vec{a})$$

by letting $X * Y$ be the jet set with carrier $UX \times UY$ such that $(x, y) \rightarrowtriangle_j (x', y')$ if either

- $x \rightarrowtriangle_j x'$ and $y = y'$,
- $x = x'$ and $y \rightarrowtriangle_j y'$.

The action on morphisms is of course faithfully inherited from the cartesian product functor on Set, which indeed produces jet set morphisms between separated products.

Definition 1.4.4°2. Given masks $\vec{a}$ and $\vec{b}$ of equal length, if $\vec{a} \sqcap \vec{b} = \vec{\bigcirc}$, i.e. if for every i we have $a_i = \bigcirc$ and/or $b_i = \bigcirc$, then we define the **semisymmetric separated product (SSS-product)** functor

$$\sqcup \S \sqcup : \mathsf{JetCubeConv}^{\maltese}_M(\mathsf{fbe}, \vec{a}) \times \mathsf{JetCubeConv}^{\maltese}_M\left(\mathsf{fbe}, \vec{b}\right) \to \mathsf{JetCubeConv}^{\maltese}_M\left(\mathsf{fbe}, \vec{a} \sqcup \vec{b}\right)$$

as follows:

- The variables of a pair of objects (V, W) are zipped on a per-degree basis:
  - If $a_i = b_i = \bigcirc$, then we list all variables of degree i of V, followed by all variables of degree i of W, all of them typed as $(\!|\frown_i|\!)$,
  - If $a_i = \nearrow$ and $b_i = \bigcirc$, then we list all variables of degree i of W, retyped as $(\!|\leftrightarrowtriangle_i|\!)$, followed by all variables of degree i of V with their original types,
  - If $a_i = \bigcirc$ and $b_i = \nearrow$, then we list all variables of degree i of V, retyped as $(\!|\leftrightarrowtriangle_i|\!)$, followed by all variables of degree i of W with their original types.

  Corollary 1.4.4°3. In $\mathsf{Cube}^{\maltese}_M$, we have

  $$\begin{aligned} \lfloor V \S W \rfloor &\cong \lfloor V \rfloor * \lfloor W \rfloor && \text{if } \maltese = \square, \\ \lfloor V \S W \rfloor &\cong \lfloor V \rfloor \times \lfloor W \rfloor && \text{if } \maltese = \boxslash. \end{aligned}$$

  Corollary 1.4.4°4. In $\mathsf{JetSet}(\vec{a} \sqcup \vec{b})$, we have

  $$\mathsf{JEP}(V \S W) \cong \mathsf{USym}_{\vec{a} \sqsubseteq \vec{a} \sqcup \vec{b}}(V) * \mathsf{USym}_{\vec{b} \sqsubseteq \vec{a} \sqcup \vec{b}}(W),$$

  where $\mathsf{USym}_{\vec{x} \sqsubseteq \vec{y}} : \mathsf{JetSet}(\vec{x}) \to \mathsf{JetSet}(\vec{y})$ is the forgetful functor.

- Recalling definition 1.4.2°8, the action of morphisms is established as follows:
  - At the level of $\mathsf{Cube}^{\maltese}_M$, by relying on functoriality of the separated/cartesian product,
  - At the level of $\mathsf{JetSet}(\vec{a} \sqcup \vec{b})$, by relying on functoriality of the separated product,
  - At the level of Set, both of these approaches reduce to functoriality of the cartesian product.

By corollary 1.4.3°2, the SSS-product extends to non-conventional jet cubes.

Definition 1.4.4°5. Given a fixed length ℓ, which we assume clear from the context, and a degree $0 \leq i < \ell$, we define the **punch mask** $\vec{\delta}^i$ by $\vec{\delta}^i_j = \bigcirc$ if $i \neq j$ and $\vec{\delta}^i_i = \nearrow$.

Thus, $\bigsqcup_i \vec{\delta}^i = \vec{\nearrow}$, and more generally $\vec{a} = \bigsqcup_i (\vec{\delta}^i \sqcap \vec{a})$.

Theorem 1.4.4°6 (SSS-factorization)**.** Let $\vec{a}$ be a mask of length ℓ. For any jet cube morphism $\hat{\varphi} : V \to W$ in $\mathsf{JetCubeConv}^{\square}_M(\mathsf{fbe}, \vec{a})$, define jet cubes $(V_i)_{0 \leq i < \ell}$ and $(W_i)_{0 \leq i < \ell}$ of mask $\vec{a} \sqcap \vec{\delta}^i$ by

- W_i consists of all variables of W of degree i, with their original typing,

- V_i as the jet cube consisting of all variables $\mathbf{j}$ of V_i such that there is a variable $\mathbf{i} \in W_i$ such that $\mathbf{i}\langle\varphi\rangle$ depends on $\mathbf{j}$. Variables of degree i are kept with their original typing, unless a variable of the same degree to their right has been used at a lower (stronger) degree or is not used at all, in which case they are retyped as $(\!| \leftrightarrow_i |\!)$. Variables of higher (weaker) degree $j > i$ are retyped as $(\!| \frown_j |\!)$. Variables of lower degree cannot occur.

Then there are jet cube morphisms $\varphi_i : V_i \to W_i$ and jet cube morphisms ρ_0 and ρ_1 that erase to cube renamings[11], such that φ factorizes as:

$$V \xrightarrow{\rho_0} \prod_i^{\S} V_i \xrightarrow{\prod_i^{\S} \varphi_i} \prod_i^{\S} W_i \xrightarrow[\cong]{\rho_1} W,$$

where $\prod^{\S}$ denotes a semisymmetric separated product.

Proof. It is immediately clear that $\prod_i^{\S} W_i \cong W$ by an isomorphism ρ_1 that erases to a renaming.[12]

We then define ρ_0 as the cube morphism that discards all variables unused by φ, and φ_i as the cube morphism such that $\mathbf{i}\langle\varphi_i \circ \rho_0\rangle = \mathbf{i}\langle\varphi\rangle$ for every variable $\mathbf{i}$ in W_i. It is then immediately clear that $\varphi = \rho_1 \circ \left(\prod_i^{\S} \varphi_i\right) \circ \rho_0$. What remains to be proven is that ρ_0 and φ_i are jet cube morphisms.

In the case of ρ_0, this is relatively trivial: we are promoting an initial segment of the variables of every degree to equijet variables of the same degree, and then projecting away some of these.

In the case of φ_i, this follows essentially from lemma 1.4.3°21. $\square$

1.4.5 Comparison to the Literature

1.4.5 (a) Point category

Proposition 1.4.5°1. The point category (terminal category) is isomorphic to $\mathsf{JetCube}^{\#}_M(\omega, [])$.

Proof. It is clear that $()$ is the only object. The only endomorphism of $()$ in $\mathsf{Cube}^{\#}_M$ is the identity, and $\lfloor \sqcup \rfloor$ is faithful, so there is only one morphism. $\square$

1.4.5 (b) Affine Symmetric Cubes

I am unsure whether the category of affine symmetric cubes $\mathsf{Cube}^{\square}_{\mathsf{IPt}_2}$ appears anywhere.

Proposition 1.4.5°2. The category $\mathsf{Cube}^{\square}_{\mathsf{IPt}_2}$ is isomorphic to $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\omega, [\bigcirc])$ and $\mathsf{JetCube}^{\boxslash}_{\mathsf{IPt}_2}(\omega, [\bigcirc])$.

Proof. The orientation set ω does not matter as all degrees are symmetric. The latter two categories are isomorphic by theorem 1.4.2°15. It is clear that $\lfloor \sqcup \rfloor : \mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\omega, [\bigcirc]) \to \mathsf{Cube}^{\square}_{\mathsf{IPt}_2}$ is bijective on objects. It is faithful, because $U : \mathsf{JetSet}([\bigcirc]) \to \mathsf{Set}$ is faithful. It is full, because any morphism can be derived in the calculus (fig. 1.1), as can be shown by induction on the length of the codomain. $\square$

1.4.5 (c) Affine Cubes

The category of affine cubes $\mathsf{Cube}^{\square}_{\mathsf{Pt}_2}$ appears in a cubical model of HoTT [BCH14] and its unary analogue in a cubical model of parametricity [BCM15].

Proposition 1.4.5°3. The category $\mathsf{Cube}^{\square}_{\mathsf{Pt}_2}$ is isomorphic to $\mathsf{JetCube}^{\square}_{\mathsf{Pt}_2}(\omega, [\bigcirc])$ and $\mathsf{JetCube}^{\boxslash}_{\mathsf{Pt}_2}(\omega, [\bigcirc])$.

Proof. Each time, the category for Pt_2 is the wide[13] subcategory of the corresponding one for IPt_2 on morphisms that do not mention $\neg$, so the result follows from proposition 1.4.5°2. $\square$

[11] Morphisms in $\mathsf{Kl}(M)$ that come from Set, i.e. are not effectful or do not use the constants and operators provided by M.

[12] In fact ρ_1 is the identity because we are working with conventional cubes.

[13] Containing all objects.

1.4.5 (d) Cartesian Cubes

One might hope to retrieve other existing categories as follows:

- De Morgan cubes $\mathsf{Cube}^{\boxslash}_{\mathsf{DM}}$ [CCHM15] as $\mathsf{JetCube}^{\boxslash}_{\mathsf{DM}}(\mathsf{f}, [\bigcirc])$,
- Cartesian cubes $\mathsf{Cube}^{\boxslash}_{\mathsf{Pt}_2}$ as $\mathsf{JetCube}^{\boxslash}_{\mathsf{DM}}(\mathsf{f}, [\bigcirc])$,
- Depth n cubes [ND18, Nuy18] as $\mathsf{JetCube}^{\boxslash}_{\mathsf{Pt}_2}(\mathsf{f}, [\bigcirc]^n)$, where $\vec{x}^n$ denotes the n-fold repetition of the list $\vec{x}$,
- ...

However, one cannot:

Proposition 1.4.5°4. The morphism $(\mathbf{i}/\mathbf{j}, \mathbf{i}/\mathbf{k}) : (\mathbf{i} : \mathbb{I}) \to (\mathbf{j} : \mathbb{I}, \mathbf{k} : \mathbb{I})$ in $\mathsf{Cube}^{\boxslash}_{M}$ is not the erasure of any jet cube morphism.

Proof. Jet sets obtained from JEP do not have diagonals. □

1.4.5 (e) Affine Depth n Cubes

Definition 1.4.5°5. The category $\mathsf{DCube}^{\diamond}_{M}(n)$ has:

- As objects lists of the form $W = (\mathbf{i}_1 : \llparenthesis k_1 \rrparenthesis, \ldots, \mathbf{i}_m : \llparenthesis k_m \rrparenthesis)$ where the $\mathbf{i}_i$ are regarded as bound de Bruijn indices and the k_i are in $\{0, \ldots, n-1\}$. We define its erasure as $\lfloor W \rfloor = (\mathbf{i}_1 : \mathbb{I}, \ldots, \mathbf{i}_m : \mathbb{I})$.
- As morphisms $\hat{\varphi} : V \to W$, morphisms $\varphi : \lfloor V \rfloor \to \lfloor W \rfloor$ such that for each $\mathbf{i} : \llparenthesis k \rrparenthesis$ in W, the expression $\mathbf{i}\langle\varphi\rangle$ mentions only variables $\mathbf{j} : \llparenthesis \ell \rrparenthesis$ in V such that $\ell \geq k$.

Clearly this category comes with a faithful functor $\lfloor \sqcup \rfloor : \mathsf{DCube}^{\diamond}_{M}(n) \to \mathsf{Cube}^{\diamond}_{M}$.

The categories $\mathsf{DCube}^{\boxslash}_{\mathsf{Pt}_2}(n)$ appear in the model of Degrees of Relatedness [ND18, Nuy18].

Proposition 1.4.5°6. The category $\mathsf{DCube}^{\square}_{\mathsf{IPt}_2}(n)$ is isomorphic to the category $\mathsf{JetCube}^{\diamond}_{\mathsf{IPt}_2}(\mathsf{f}, [\bigcirc]^n)$ for $\diamond \in \{\boxslash, \square\}$.

Proof. By theorem 1.4.2°15, the value of $\diamond$ does not matter, so let us set $\diamond = \square$. We construct a functor $F : \mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{f}, [\bigcirc]^n) \to \mathsf{DCube}^{\square}_{\mathsf{IPt}_2}(n)$ such that $\lfloor \sqcup \rfloor \circ F = \lfloor \sqcup \rfloor$:

- $F(\mathbf{i}_1 : \llparenthesis \frown_{k_1} \rrparenthesis, \ldots, \mathbf{i}_m : \llparenthesis \frown_{k_m} \rrparenthesis) = (\mathbf{i}_1 : \llparenthesis k_1 \rrparenthesis, \ldots, \mathbf{i}_m : \llparenthesis k_m \rrparenthesis)$,
- For the action on morphisms, we have nothing to choose, we can only verify that it exists. This is done by induction on the derivation in the calculus (fig. 1.1).

It is clear that F is bijective on objects, and faithful. Fullness is proven by proving by induction on the length of the codomain that every morphism of depth n cubes can be derived in the calculus. □

Proposition 1.4.5°7. The category $\mathsf{DCube}^{\square}_{\mathsf{Pt}_2}(n)$ is isomorphic to the category $\mathsf{JetCube}^{\diamond}_{\mathsf{Pt}_2}(\mathsf{f}, [\bigcirc]^n)$ for $\diamond \in \{\boxslash, \square\}$.

Proof. Each time, the category for Pt_2 is the wide subcategory of the corresponding one for IPt_2 on morphisms that do not mention $\neg$, so the result follows from proposition 1.4.5°6. □

1.4.5 (f) Comparison to Pinyo and Kraus's Twisted Cube Category

In this section, we relate jet cubes to Pinyo and Kraus's twisted cubes [PK19] when $\vec{a} = [\nearrow]$. $\mathsf{JetSet}([\nearrow])$ is the category of proof-irrelevant reflexive graphs. Pinyo and Kraus use arbitrary proof-irrelevant graphs, but since $\top$ is reflexive and the twisted prism functor [PK19, def. 4] restricts to reflexive graphs, all twisted cubes are reflexive graphs anyway.

Two twisted cube categories appear (up to isomorphism) in [PK19], and we show that we can recover both.

Definition 1.4.5°8. [PK19, def. 25] The category $\mathsf{TwCube}_{\mathsf{graph}}$ has as objects [$\nearrow$]-jet-cubes (i.e. natural numbers) and as morphisms $V \to W$ all jet set morphisms (i.e. graph morphisms) $\mathsf{JEP}(V) \to \mathsf{JEP}(W)$.

Proposition 1.4.5°9. $\mathsf{TwCube}_{\mathsf{graph}}$ is isomorphic to $\mathsf{JetCube}^{\boxtimes}_{\mathsf{Boo}}(\mathsf{f}, [\nearrow])$.

Proof. Clearly, the objects correspond. The morphisms $V \to W$ of $\mathsf{JetCube}^{\boxtimes}_{\mathsf{Boo}}(\mathsf{f}, [\nearrow])$ are morphisms $f : \mathsf{JEP}(V) \to \mathsf{JEP}(W)$ such that $Uf : \mathsf{EP}(\lfloor V \rfloor) \to \mathsf{EP}(\lfloor W \rfloor)$ lifts to a morphism of cubes, which it always does by proposition 1.4.1°7. $\square$

We thoroughly rephrase Pinyo and Kraus's ternary twisted cube category:

Definition 1.4.5°10. [PK19, def. 34] The category $\mathsf{TwCube}_{\mathsf{tri}}$ has as objects [$\nearrow$]-jet-cubes (i.e. natural numbers) and as morphisms $V \to W$ all jet set morphisms (i.e. graph morphisms) $\mathsf{JEP}(V) \to \mathsf{JEP}(W)$ or, equivalently by the previous proposition, all jet cube morphisms $V \to W$ in $\mathsf{JetCube}^{\boxtimes}_{\mathsf{Boo}}(\mathsf{f}, [\nearrow])$, generated by the rules TERMINAL, [SRC:FWD immediately below INV:BCK], TGT:FWD and PRISM:FWD in fig. 1.1.

The shared reader may object that Pinyo and Kraus define the morphisms of $\mathsf{TwCube}_{\mathsf{tri}}$ by *constructing* them inductively, rather than by selecting them inductively as we do above. However:

Corollary 1.4.5°11. Any morphism of $\mathsf{TwCube}_{\mathsf{tri}}$ as defined here, has a unique derivation using the given rules. $\square$

Proposition 1.4.5°12. $\mathsf{TwCube}_{\mathsf{tri}}$ is isomorphic to $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{f}, [\nearrow])$.

Proof. Since the rules mentioned in definition 1.4.5°10 pertain to the calculus of $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{fbe}, [\nearrow])$ and in fact always yield premises/conclusions about f-jet-cubes for conclusions/premises about f-jet-cubes, and since $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{f}, [\nearrow])$ is a faithful subcategory of $\mathsf{JetCube}^{\boxtimes}_{\mathsf{Boo}}(\mathsf{f}, [\nearrow])$ by proposition 1.4.1°8, it is clear that the identity-on-objects functor $\mathsf{TwCube}_{\mathsf{tri}} \to \mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{f}, [\nearrow])$ is faithful, i.e. we can think of $\mathsf{TwCube}_{\mathsf{tri}}$ as a wide subcategory of $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{f}, [\nearrow])$.

To show fullness, we note the following facts about the calculus for $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{fbe}, [\nearrow])$:

1. No rule (read bottom-up) introduces equijets in the codomain.
2. In absence of equijets in the codomain, no rule changes the mask (except SYMMETRIZE applied to the terminal codomain, but then apply TERMINAL instead).
   - In particular, SYMMETRIZE is useless.
3. Equijet variables can only be used at strictly stronger (lower) degrees, or at the current degree i if $a_i = \bigcirc$. Since we have only one directed degree, equijet variables cannot be used.
   - Hence WKN can only be used to derive constant morphisms, which can instead be derived by [SRC:FWD after INV:BCK] and TGT:FWD.
   - Hence EXCHANGE is useless.
4. At mask [$\nearrow$], the rule CONCURSOR cannot be used as there is only one degree.
5. Since INV:FWD and INV:BCK are mutually inverse, they can together be freely inserted everywhere. Hence, we can replace the rule SRC:FWD with [SRC:FWD after INV:BCK].
6. INV:FWD and INV:BCK can be pushed up through any of the remaining rules except PRISM:FWD and PRISM:BCK. Thus, we only involute right before using a variable. All other rules (still read bottom-up) do not turn an f-codomain into an fbe-codomain, i.e. they do not introduce opposite jets. Thus, we can assume the codomain is a forward jet cube until we encounter $\neg\mathbf{i}$.
7. This means that until we encounter $\neg\mathbf{i}$, we only need the rules TERMINAL, [SRC:FWD after INV:BCK], TGT:FWD and PRISM:FWD. These do not turn an f-codomain into an fbe-codomain. Thus, until we encounter $\neg\mathbf{i}$, we can assume the domain is an f-jet-cube.
8. If both domain and codomain are forward, we cannot encounter $\neg\mathbf{i}$.

This means we can always rewrite a derivation tree in the calculus for $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{fbe}, [\nearrow])$ of a morphism in $\mathsf{JetCube}^{\square}_{\mathsf{IPt}_2}(\mathsf{f}, [\nearrow])$ to use only the prescribed rules. $\square$

Chapter 2

Modalities

Chapter 3

Paths and Bridges

Remark 3.0.0°1. **Move this remark** □

We note that it is always possible to restrict our mode theory, by discarding modes but keeping the same modalities and 2-cells between remaining modes. We could decide to restrict to any of the following subsets of modes:

- Modes of the form $[\nearrow]^*$, i.e. where all degrees are polar,
- Modes of the form $([\bigcirc] | [\bigcirc, \nearrow])^*$, i.e. where we think of a level as containing a path relation and optionally a weaker jet relation,
  - Modes of the form $[\bigcirc, \nearrow]^*$ where the presence of a jet relation at each level is required,
- Modes of the form $([\bigcirc] | [\nearrow, \bigcirc])^*$, i.e. where we think of a level as containing a bridge relation and optionally a stronger jet relation,
  - Modes of the form $[\nearrow, \bigcirc]^*$ where the presence of a jet relation at each level is required,
- Modes of the form $([\bigcirc, \bigcirc] | [\bigcirc, \nearrow, \bigcirc])^*$, i.e. where we think of a level as containing a path relation and a weaker bridge relation and optionally, in between, a jet relation,
  - Modes of the form $[\bigcirc, \nearrow, \bigcirc]^*$ where the presence of a jet relation at each level is required.

We will occasionally discuss these subtheories. By considering all of $\mathsf{List}\,\mathbb{A}$ in the current paper, we maintain generality.

Relate cubes with and without equijets. □

Chapter 4

Transpension

Does $\mathrm{Tw} : \mathrm{sSet} \to \mathrm{sSet}$ have a transp type? (Yes if it's $\mathrm{Tw}_!$) □

Appendix A

Version history

v0.1 Wrote chapter 1.
v0.1.1 Changed title.
Proofread chapter 1.
Added missing clause for CONCURSOR in proof of theorem 1.4.3°4.
Cleaned up affine boolean normalization lemmas in section 1.4.3 (b).
Introduced orientation sets (definition 1.4.2°1).
Introduced conventional cubes (definition 1.4.3°1).
General cleanup.

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